Infrageometry: Geometry Emerging from Hypergraph Rewriting

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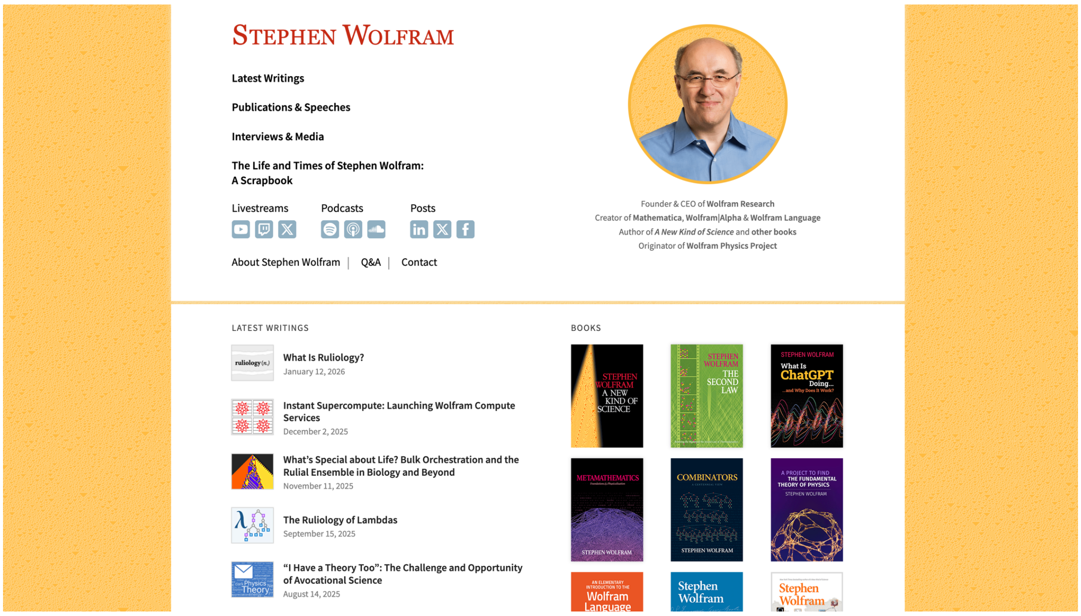
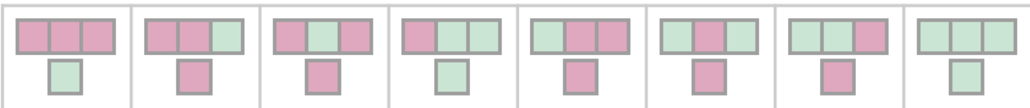
Madrid, February 20, 2026

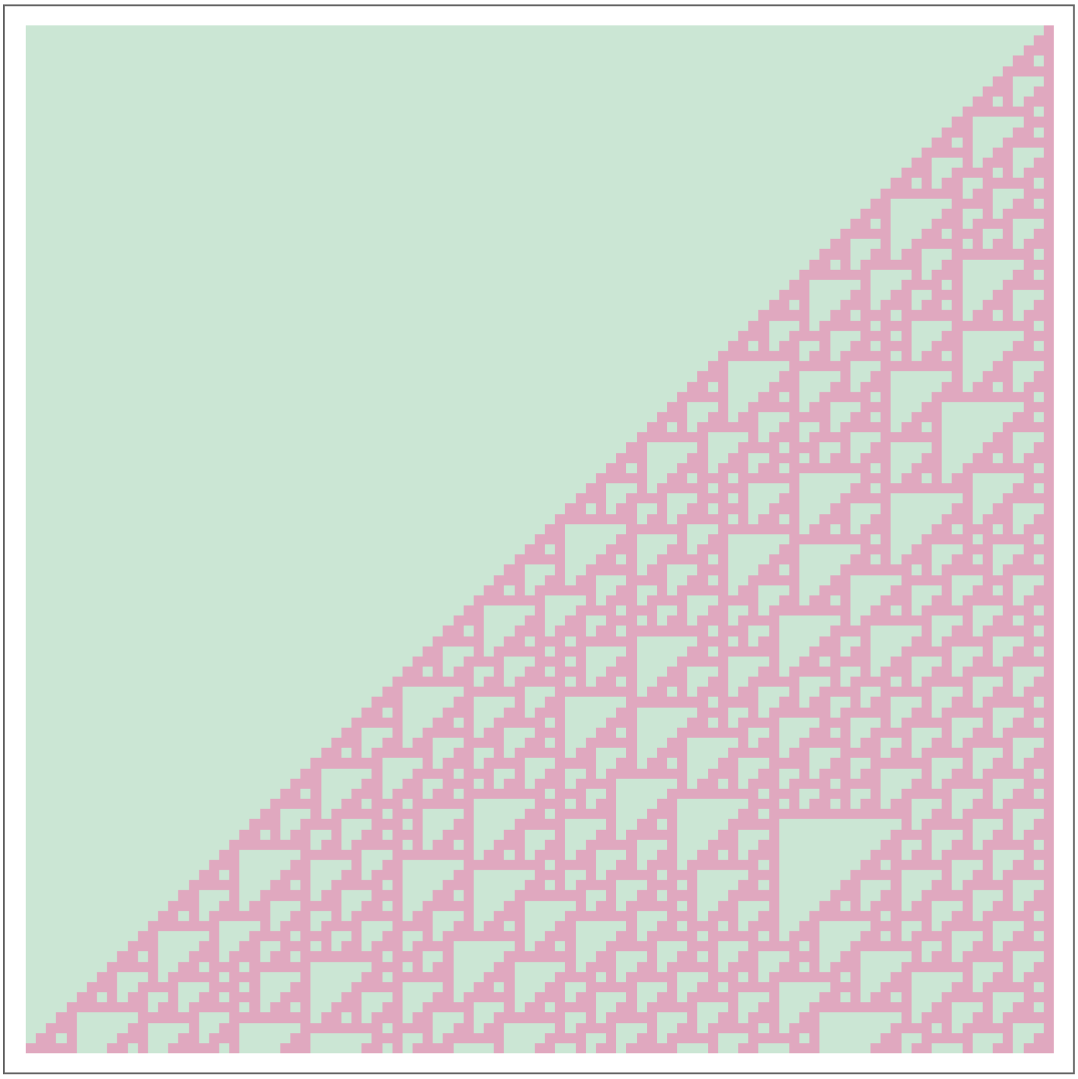
Setting the Stage: Wolfram Physics Project

- **Stephen Wolfram**, in his study of *cellular automata*, became fascinated by **complex systems**, **emergent structures**, and **computational irreducibility**.

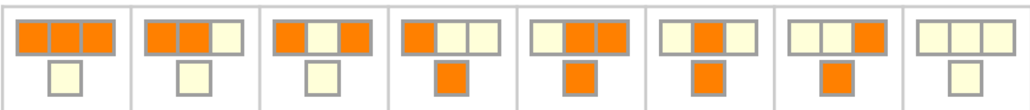


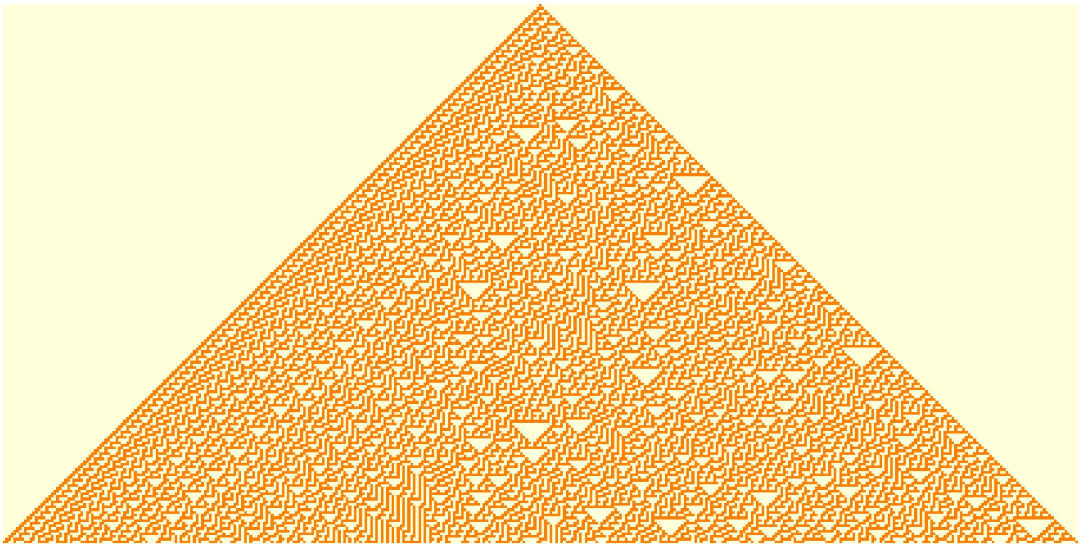
- **Rule 110** proven to be Turing complete:





- **Rule 30** conjectured to be computationally irreducible (≠undecidable) :





- **Can the entire universe emerge as a "thermolimit" of a simple computational model such as hypergraph rewriting?**

In[ ]:=
```
Import[NotebookDirectory[] <> "rewriting.mp4"]
```

Out[ ]=

- *Paradigm change:* Mainstream science relies on abstractions—such as the real line—that originate from **extremely coarse observations of an extremely complex substrate**. These abstractions enable convenient constructions, like differential equations, that are well suited for pen-and-paper reasoning but whose exact instantiations are arguably unrealistic. If we aim for a truly **fundamental theory of nature**, it may be more appropriate to begin with **minimal models on the least complex discrete objects**, which experiments suggest are more realistic. From there, we can use supercomputers and machine learning for **empirical exploration**, and rely on effective theories, coarse invariants, and statistics to make predictions for **large aggregates**, recovering familiar mathematical and physical structures only at large scales.

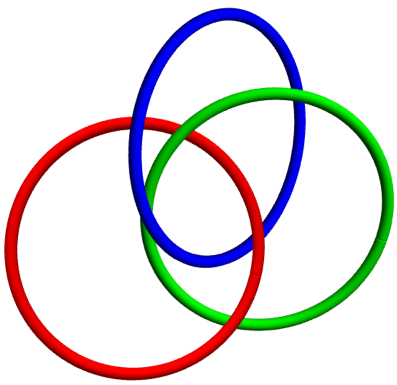
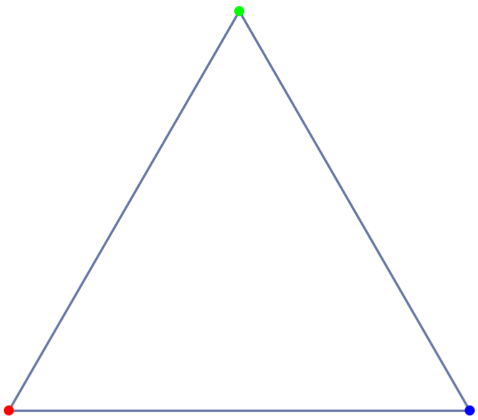
- *For orientation:* WPP argues that the **length of an edge** is far below the Planck length 1.6163×10^{-35} m, with a working estimate around 10^{-93} m. On the other hand, the smallest experimentally accessible length scales are limited by diffraction that roughly corresponds to the wavelength $\sim 10^{-7}$ m for optical devices and goes down to about $\sim 10^{-13}$ m for synchrotron X-rays (European XFEL in Hamburg).

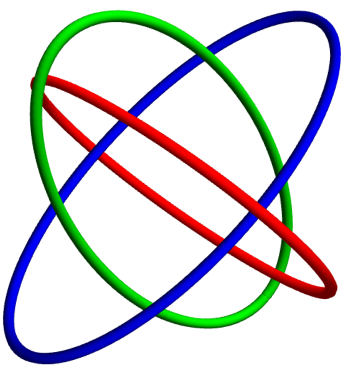
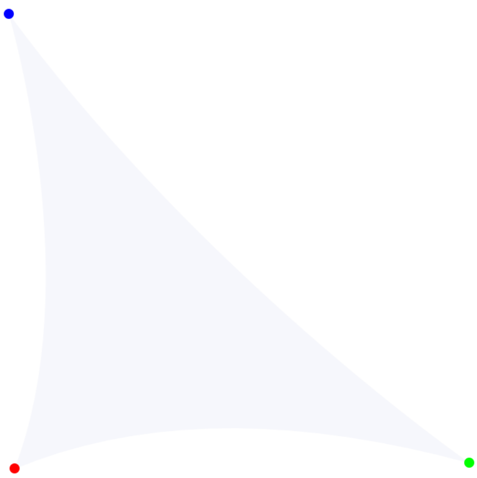
- **Infrageometry** is an attempt to define a geometrical framework for the Wolfram Physics Project.

Recollection: Hypergraphs and Hypergraph Rewriting

- A **hypergraph** is a minimal combinatorial structure that can encode relations of arbitrary arity

In[∘]:=
```
plotParametricKnots[x__] := Graphics3D[#, Boxed → False, Axes → False] &@
    (Style[Tube[Table[#[[1]][α], {α, 0, 2 Pi, 0.05}], 0.05], #[[2]]] & /@ x);
links = plotParametricKnots /@ {
     {{α ↦ {Cos[α], Sin[α], 0}, Red},
      {α ↦
        {1, .5, 0} + RotationMatrix[Pi / 10, {1, 0, 0}].{Cos[α], Sin[α], 0}, Green},
      {α ↦ {.5, 1, 0} + RotationMatrix[Pi / 5, {0, 1, 0}].{Cos[α], Sin[α], 0}, Blue}}
     ,
     {{α ↦ {2 * Cos[α], Sin[α], 0}, Red},
      {α ↦ RotationMatrix[Pi / 2, {1, 0, 0}].RotationMatrix[Pi / 2, {0, 1, 0}].
         {2 * Cos[α], Sin[α], 0}, Green}, {α ↦ RotationMatrix[Pi / 2, {0, 1, 0}].
         RotationMatrix[Pi / 2, {1, 0, 0}].{2 * Cos[α], Sin[α], 0}, Blue}
     }};
hgs = Hypergraph[#, VertexStyle → {1 → Red, 2 → Blue, 3 → Green},
      EdgeLabels → "EdgeTag", PlotTheme → Automatic] & /@ {
     {{1, 2}, {2, 3}, {3, 1}},
     {{1, 2, 3}}
    };
Column[
 Row[{#[[1]], Spacer[20], "~", Spacer[20], #[[2]]}] & /@ Transpose[{links, hgs}] ]
```
Out[∘]=

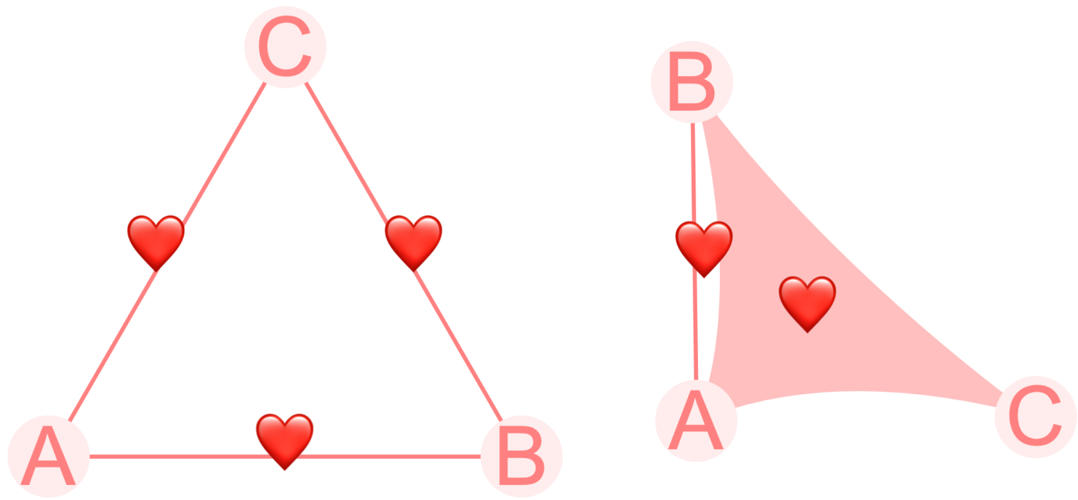
 ~ 

 ~ 

- By labeling edges and adding symmetries we can encode any relations (even “non-higher order”)

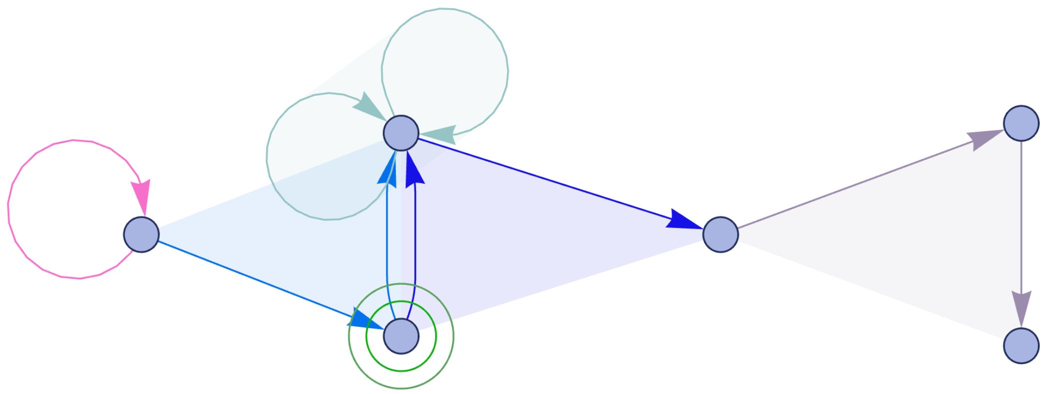
In[ ]:=
```
style = {VertexStyle → {_ → LightPink}, VertexLabels →
    {_ → Placed[Style["Name", 40], Center]}, VertexLabelStyle → {_ → Pink},
   VertexSize → {_ → 0.1}, EdgeStyle → {_ → Directive[Pink, Thick]},
   EdgeLabels → {_ → Placed[Style["  ", 30], Center]}, ImageSize → Medium};
t1 = Hypergraph[{{"A", "B"}, {"B", "C"}, {"C", "A"}},
   EdgeSymmetry → {_ → "Directed"}];
t2 = Hypergraph[{{"A", "B", "C"}, {"A", "B"}}];
style =
 Row@{Hypergraph[t1, Sequence @@ style],
   Spacer[40], Hypergraph[t2, Sequence @@ style]}
```

Out[ ]=



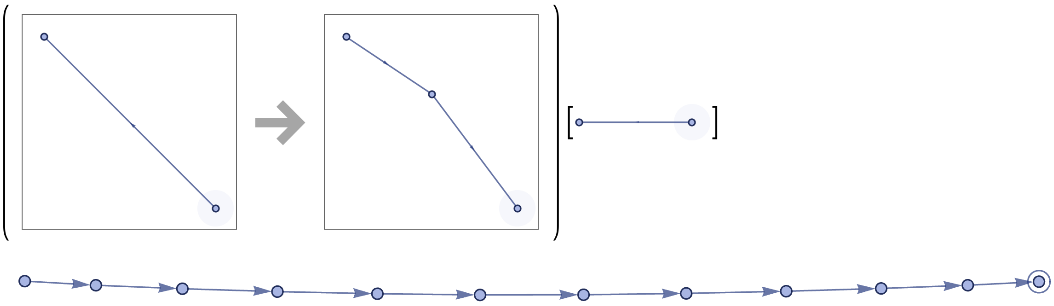
In[ ]:=
```
ResourceFunction["WolframModelPlot"][
 {{1, 2, 3}, {2, 3, 5}, {5, 7, 8}, {1, 1}, {3, 3, 3}, {2}, {2}},
 EdgeStyle → RandomColor[7]]
```

Out[ ]=



- **Hypergraph rewriting** is a form of surgery on hypergraphs.

In[ ]:=

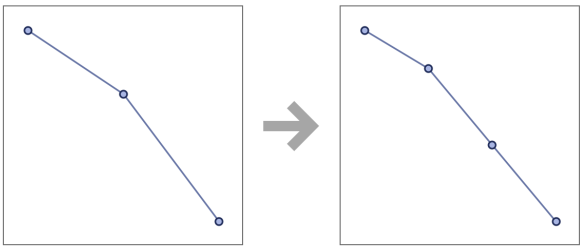


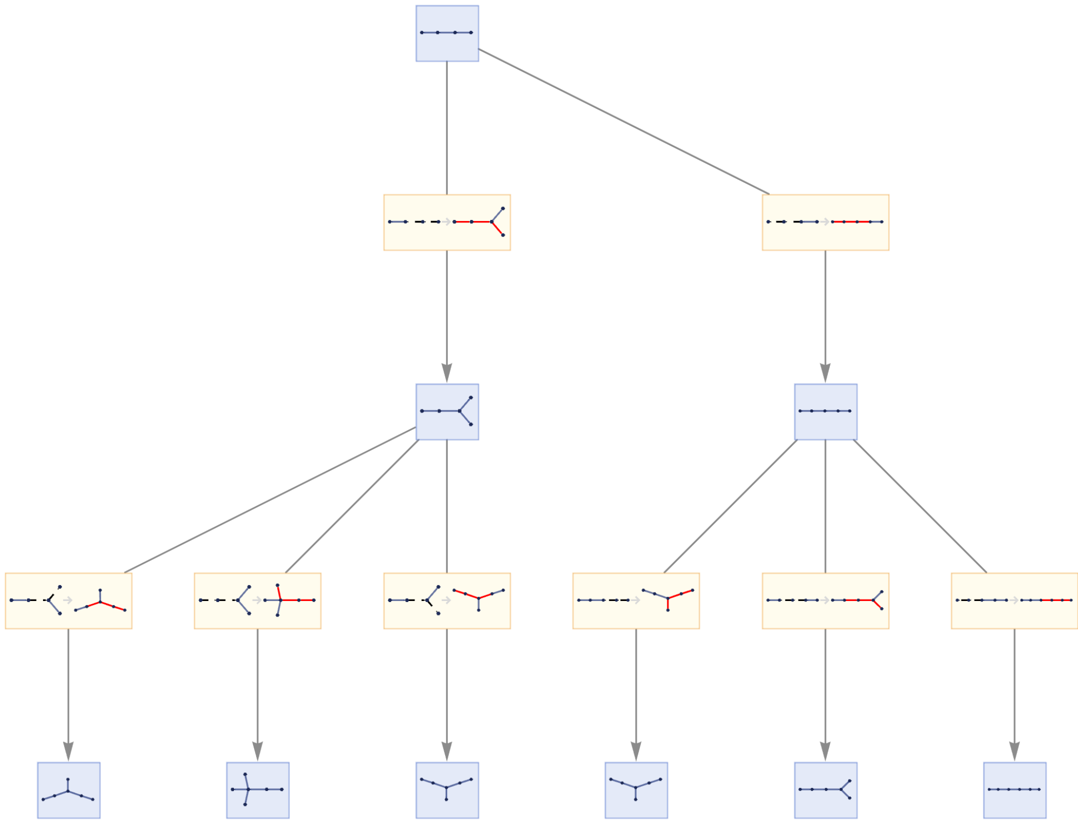
- A single rewriting rule can apply at **several possibly overlapping sites**, producing a **multicomputation tree**:

In[ ]:=

```
ms = MultiwaySystem[
   {{{1, 2}, {2, 3}} → {{1, 2}, {2, 3}, {3, 4}}}, {{{1, 2}, {2, 3}, {3, 4}}}];
Column @ { HypergraphRule[
   Hypergraph[{{1, 2}, {2, 3}}], Hypergraph[{{1, 2}, {2, 3}, {3, 4}}]],
  ms["EvolutionEventsGraph", 2], Spacer[40]}
```

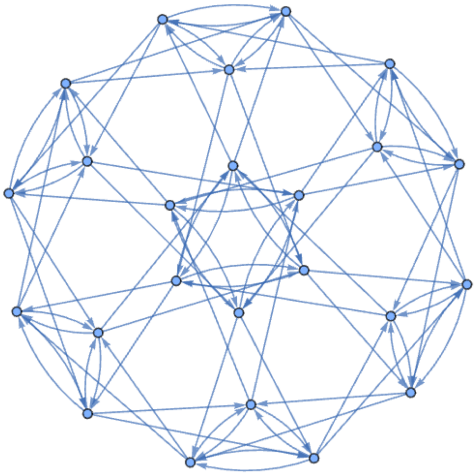
Out[ ]=

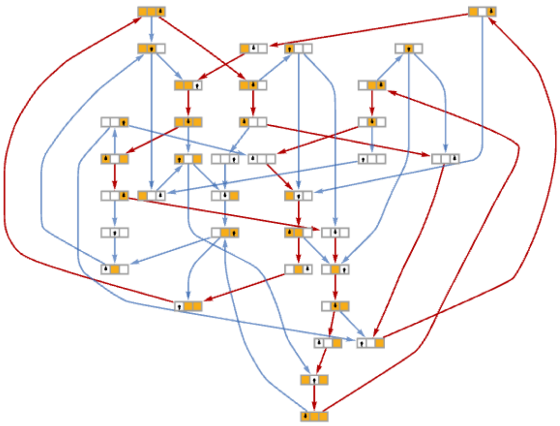


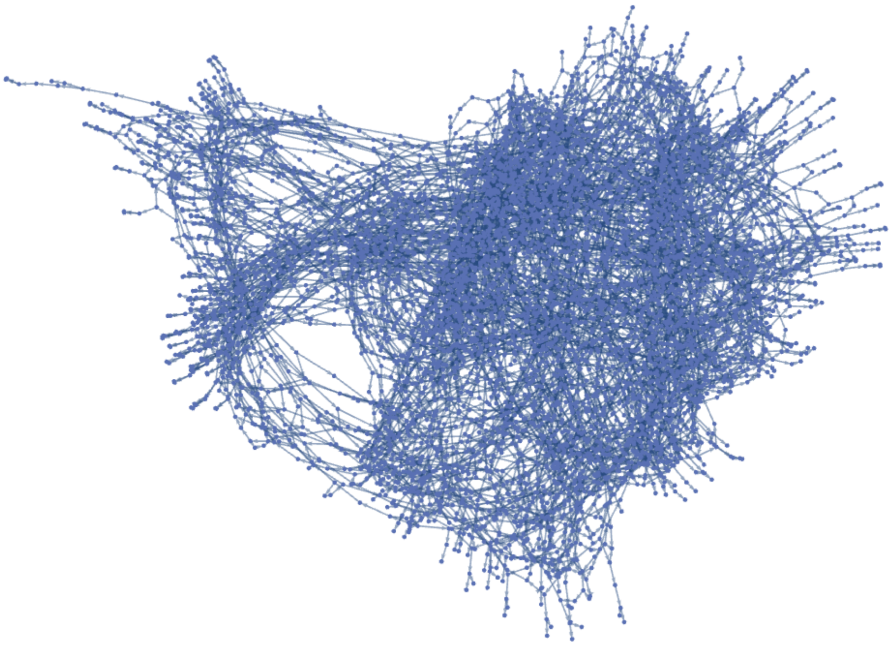


- Hypergraph rewriting is a model of computation that is **non-deterministic** and **Turing complete**.

- You can also think of the rewriting system as a **multi-vector field on the space of states** (isomorphism classes of hypergraphs), with a deterministic computation corresponding to a single vector field. For illustration, here are the **state graphs** of a *multi-way generalized shift* and the *union of (2,2)-Turing machines 2506 and 506*—firstly on a cyclic tape of length 3 with **periodic orbits** highlighted to indicate non-terminating loops and secondly on a cyclic tape of length 10 truncated at **halting states**:



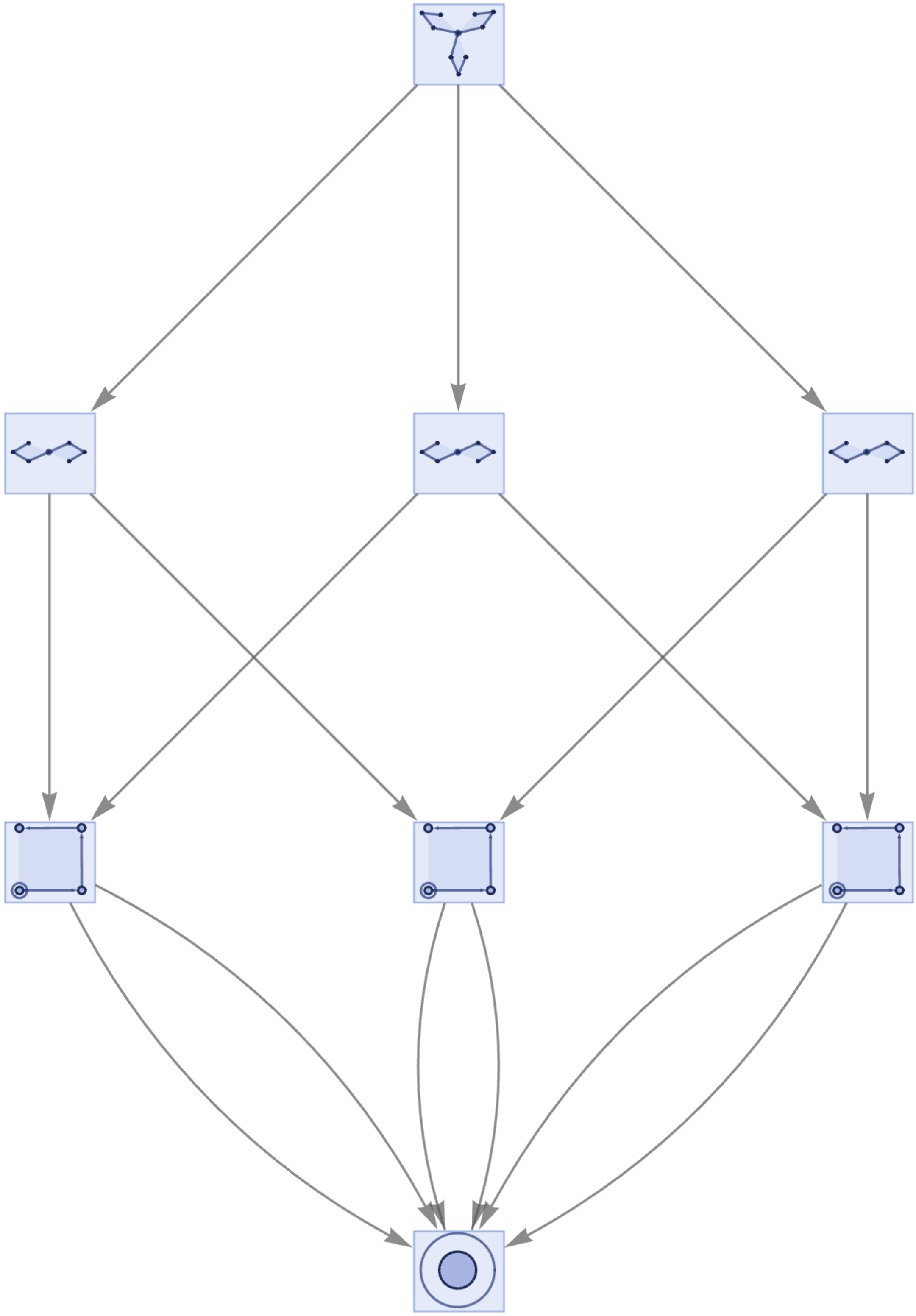




- *Note:* An important characteristic of rewriting systems is **confluence**, which ensures that different branches eventually come together. On the picture below we illustrate **terminating confluence,** which is usually associated with **shrinking rules**. However, the physics project uses **expansive rules** so confluence is more subtle.

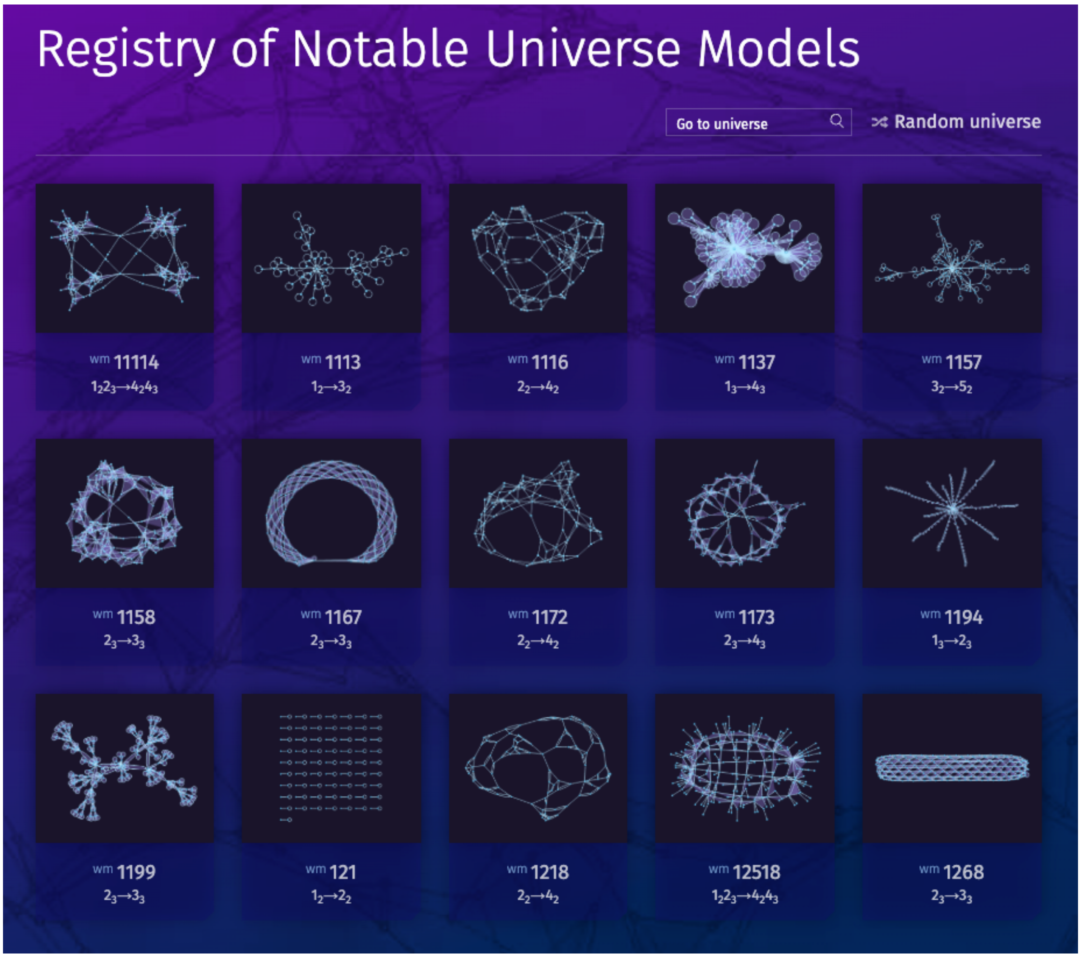
In[∘]:=
```
ms = MultiwaySystem[{{{1, 2, 3, 4}, {1}} → {{1}}},
    {{{1, 2, 3, 4}, {1, 5, 6, 7}, {1, 8, 9, 10}, {1}}}];
ms["StatesGraph", 4]
```

Out[∘]=



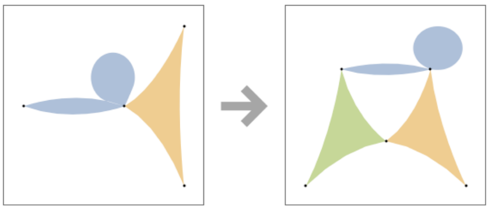
Geometry in Wolfram Physics Project

- Plot of the resulting hypergraph along a ***canonical branch*** of the rewriting system—obtained by always rewriting the least recently rewritten site—in a way that suggests an underlying geometry.

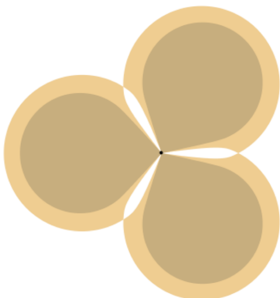


In[◦]:=
```
Column [{ Row[{
    HypergraphRule[Hypergraph[{{1, 1, 2}, {3, 4, 1}}, PlotTheme → "Colored"],
     Hypergraph[{{1, 1, 3}, {4, 5, 1}, {2, 5, 3}}, PlotTheme → "Colored"]],
    "    applied 500 times to",
    Hypergraph[{{0, 0, 0}, {0, 0, 0}}, PlotTheme → "Colored"], "  gives"
   }],
  ResourceFunction["GraphReconstructedSurface"][
   ResourceFunction["WolframModel"][
    {{1, 1, 2}, {3, 4, 1}} → {{1, 1, 3}, {4, 5, 1}, {2, 5, 3}},
    {{0, 0, 0}, {0, 0, 0}}, 500, "FinalState"] ]}, Spacings → 2]
```

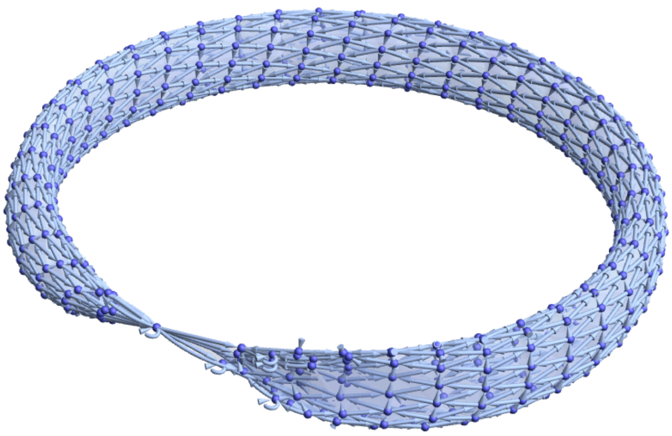
Out[◦]=



applied 500 times to



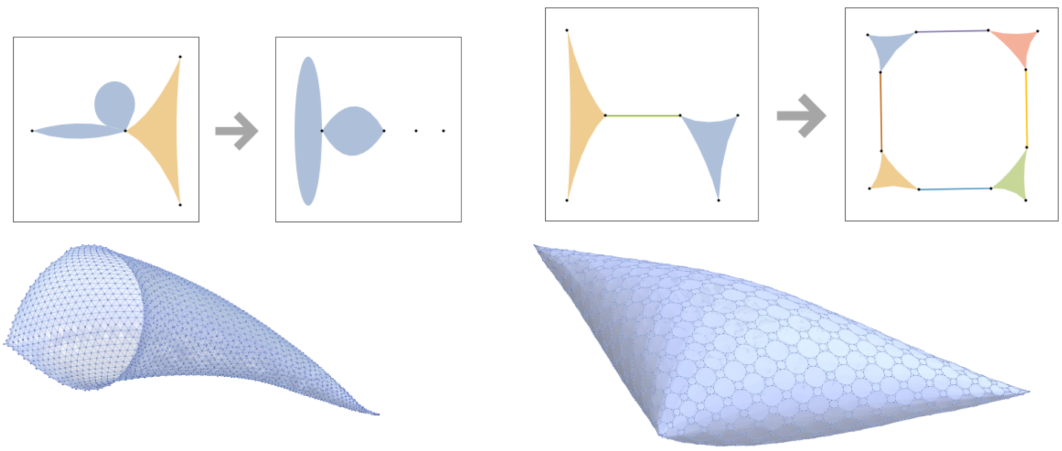
gives



- Different geometry for different rules:

In[ ]:=
```
Row[{
  Column[{
    HypergraphRule[Hypergraph[{{1, 1, 2}, {1, 3, 4}}, PlotTheme → "Colored"],
     Hypergraph[{{4, 4, 3}, {2, 5, 3}, {2, 5, 3}}, PlotTheme → "Colored"]],
    ResourceFunction["GraphReconstructedSurface"][
     WolframModel[{{1, 1, 2}, {1, 3, 4}} → {{4, 4, 3}, {2, 5, 3}, {2, 5, 3}},
      {{0, 0, 0}, {0, 0, 0}}, 2000, "FinalState"]]
   }],
  Spacer[40],
  Column[{
    HypergraphRule[
     Hypergraph[{{1, 2, 3}, {4, 5, 6}, {1, 4}}, PlotTheme → "Colored"],
     Hypergraph[{{2, 7, 8}, {3, 9, 10}, {5, 11, 12}, {6, 13, 14},
       {13, 8}, {7, 10}, {9, 12}, {11, 14}}, PlotTheme → "Colored"]],
    ResourceFunction["GraphReconstructedSurface"][
     WolframModel[{{1, 2, 3}, {4, 5, 6}, {1, 4}} → {{2, 7, 8}, {3, 9, 10},
        {5, 11, 12}, {6, 13, 14}, {13, 8}, {7, 10}, {9, 12}, {11, 14}},
      {{0, 0}, {0, 0}, {0, 0}, {0, 0, 0}, {0, 0, 0}}, 10, "FinalState"], 8]
   }]
 }]
```

Out[ ]=

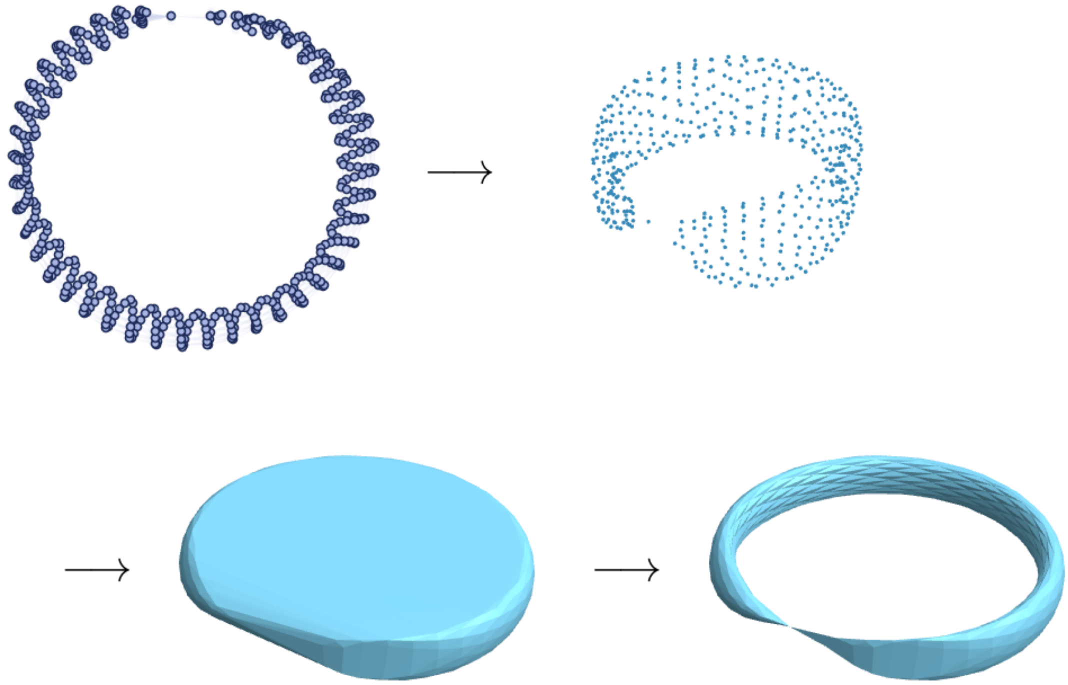


- Hypergraph → Spring Electric Embedding of 1-skeleton → Delaunay triangulation → Scale Filtering

In[ ]:=
```
hg = WolframModel[{{1, 1, 2}, {3, 4, 1}} → {{1, 1, 3}, {4, 5, 1}, {2, 5, 3}},
    {{0, 0, 0}, {0, 0, 0}}, 500, "FinalState"] ;
g = ResourceFunction["HypergraphToGraph"] [ hg ];
ge = GraphEmbedding[g, "SpringElectricalEmbedding", 3];
d = DelaunayMesh[ge];
m = ResourceFunction["NonConvexHullMesh"] [ge, 2.5];

Row[{Hypergraph[hg], Spacer[20], ⟶, Spacer[20],
  ListPointPlot3D[ge, Axes → False, Boxed → False], Spacer[20],
  ⟶, Spacer[20], d, Spacer[20], ⟶, Spacer[20], m}]
```

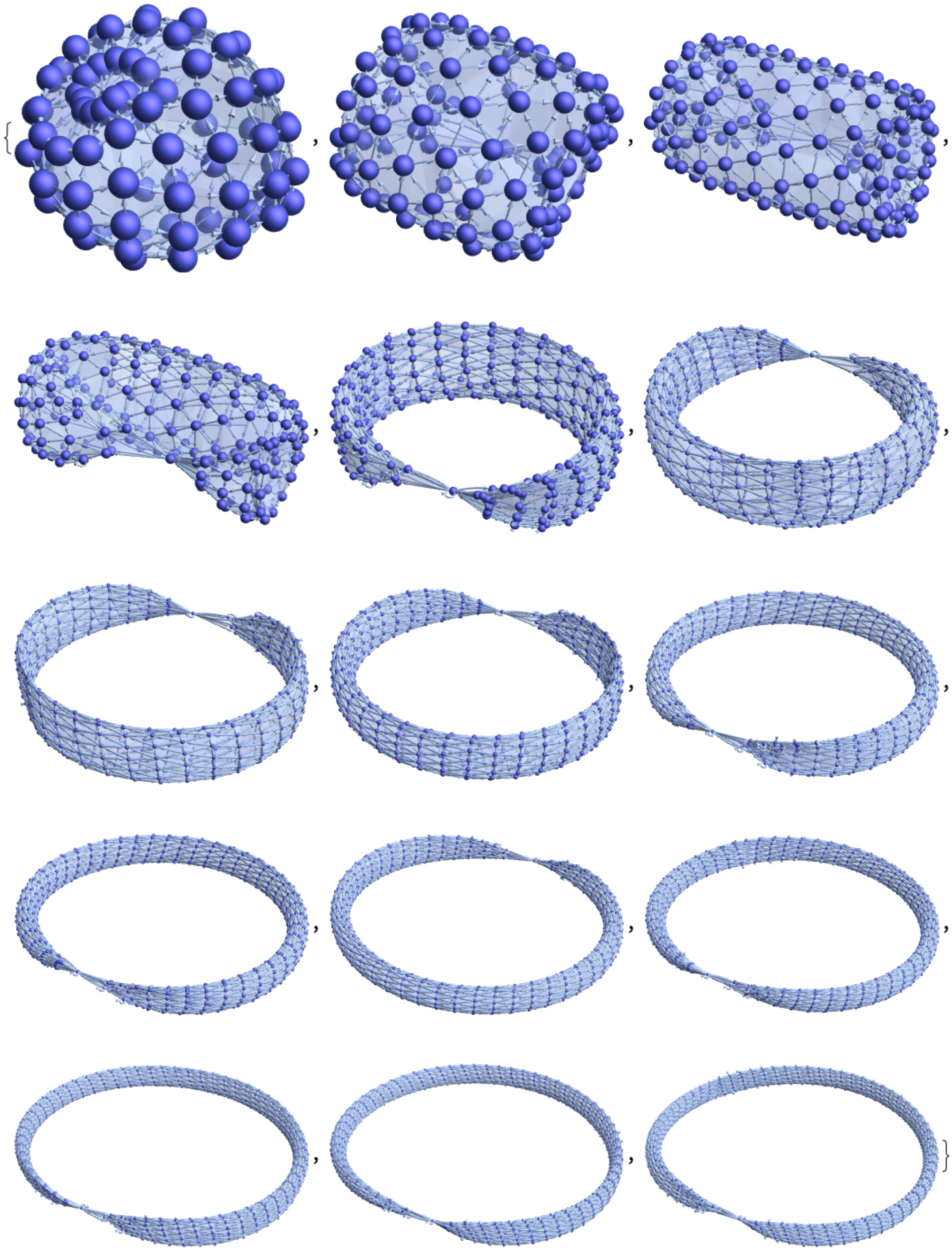
Out[ ]=



- You can “verify” that with each additional rewriting step the shape “converges”:

In[ ]:=
```
Table[
 ResourceFunction["GraphReconstructedSurface"][
  ResourceFunction["WolframModel"][{{1, 1, 2}, {3, 4, 1}} →
    {{1, 1, 3}, {4, 5, 1}, {2, 5, 3}}, {{0, 0, 0}, {0, 0, 0}}, n, "FinalState"]],
 {n, 100, 800, 50}]
```

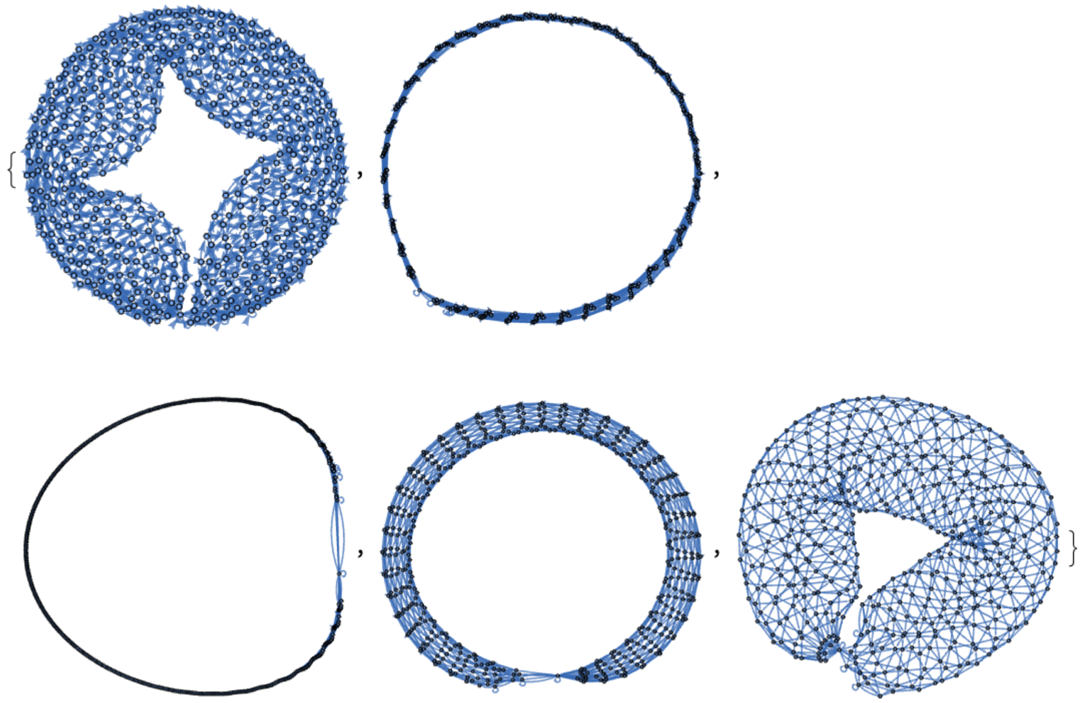
Out[ ]=



- An embedding determines the **extrinsic geometry** of a hypergraph. There are several possible "optimization embeddings", each producing a slightly different geometry. Infrageometry aims to describe the **intrinsic infrageometry** of the hypergraph itself (give by adjacency tensor), and to study what of its properties are preserved by embeddings.

In[∘]:=
```
Table[Graph[g, GraphLayout → emb],
 {emb, {"GravityEmbedding", "HighDimensionalEmbedding",
   "SpectralEmbedding", "SpringElectricalEmbedding", "SpringEmbedding"}
}]
```

Out[∘]=



- Perhaps suitable frameworks already exist—for example, connections to **coarse geometry**, **Gromov–Hausdorff distance**, **Gromov's quasi-isometries**, **spin networks**, ... But we want to arrive at these structures in the Wolfram framework in the context of hypergraph rewriting and multicomputation and later build **bridges** between these approaches.

Infrageometry

- Study of **intrinsic geometry of hypergraphs** arising in hypergraph rewriting.
  - What **geometric minimal models** and **observables** make sense for a graph?
  - **Emergence and persistence** of properties and of relations between observables in hypergraph rewriting.
  - Dependence of geometry on the computational branch and on the rewriting rules—**Ruliology**.
  - Design of tests of "**geometricity**" and **adaptive evolution** (pruning "non-geometric branches").
  - Relations to properties of embeddings.
  - What **graph substrates** support what constructions and theories?
- Infrageometry can also be viewed as a **geometric reconstruction**—an inverse to **discretization**—where a "small" hypergraph can represent multiple geometries, and successive rewritings into larger hypergraphs progressively specify the geometry more precisely. This corresponds to experiment, since objects at quantum scales are not supposed to possess definite geometries themselves but instead have the potential to organize into geometric structure at macroscopic scales.

Main Goals

- Discrete notion of dynamics in terms of rewriting systems, obtaining solutions of ODEs, in particular to the Einstein equations or the Navier–Stokes equations, in the thermolimit.
- Framework for quantum gravity.

Current Projects

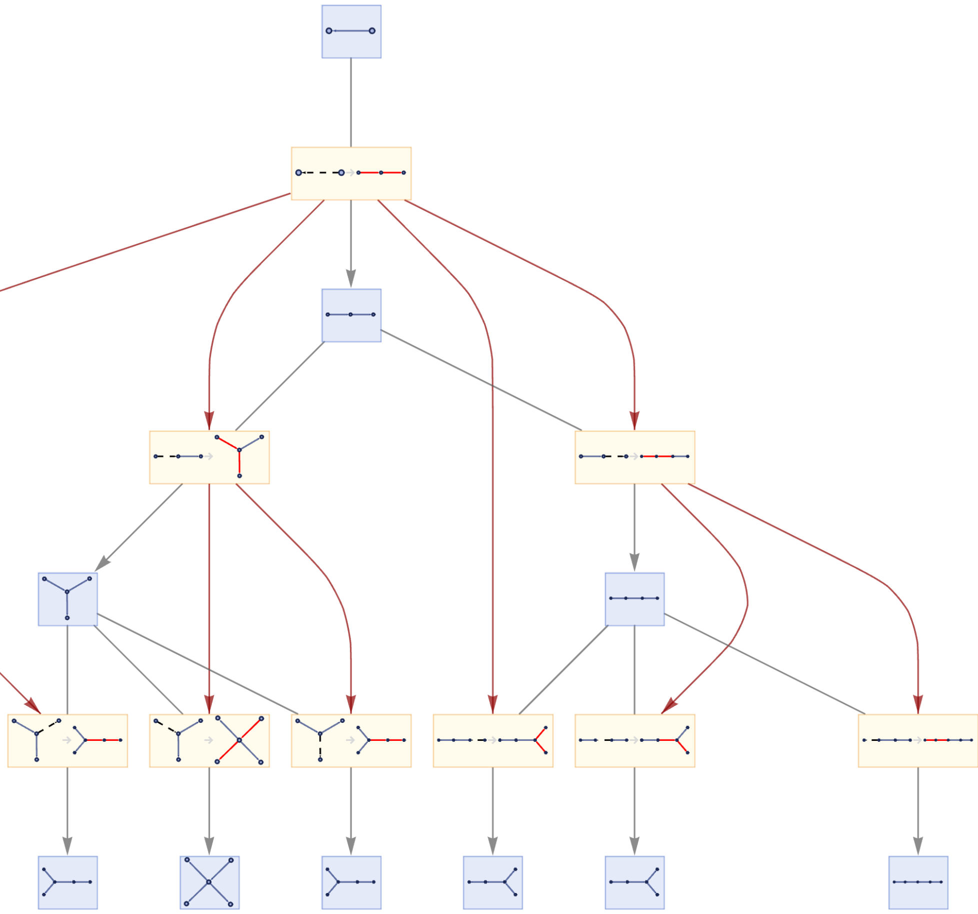
- **Synthetic Infrageometry**—an axiomatic approach to the construction of lines, subspaces, circles, and related objects, analogous to Euclidean geometry.
- **Infracausality**—axiomatic approach to special relativity derived from causal graphs.
- **Infragaugetheory**—minimal models of fibrations, moduli of connections, dynamics, and quantum corrections to Wilson loops.
- **Infraanalysis**—multi-variate analysis on graphs *coordinatized* by collections of directed acyclic subgraphs with built-in renormalization.

Infrageometry from Multicomputation

- Discrete world is pervaded by ambiguities, and the **multicomputation framework** attempts to organize them systematically and extract useful information.

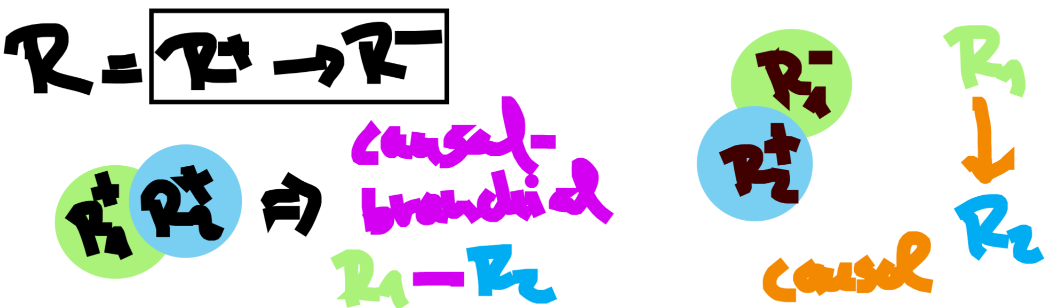
In[ ]:=
```
ms = MultiwaySystem[{{{1, 2}} → {{1, 2}, {2, 3}}}, {{{1, 2}}}];
ms["EvolutionCausalGraph", 3]
```

Out[ ]=



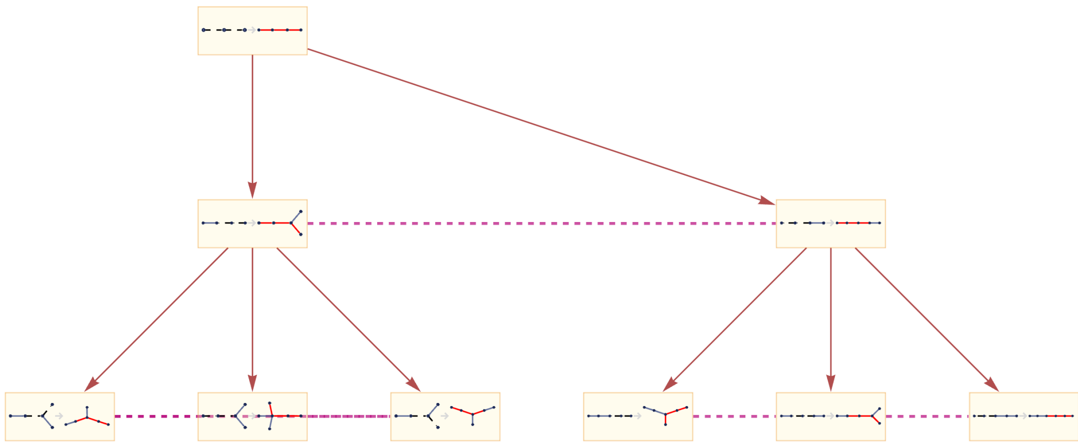
Infrageometry of causal-branchial graphs

- From the *multicomputation tree of hypergraph rewriting* we extract **causal** and **branchial** relations between rewriting events, leading to the notion of **causal-branchial graph.**

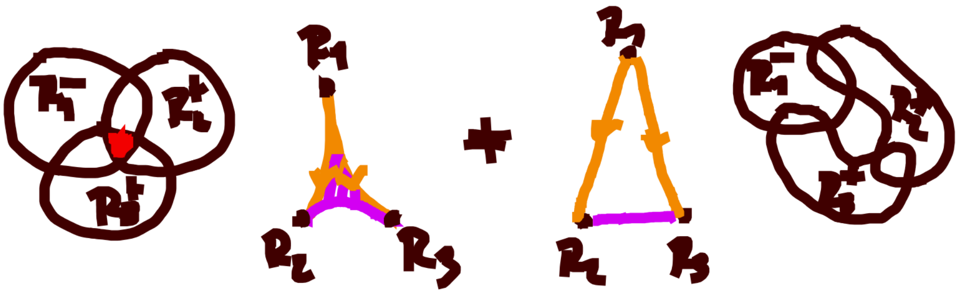


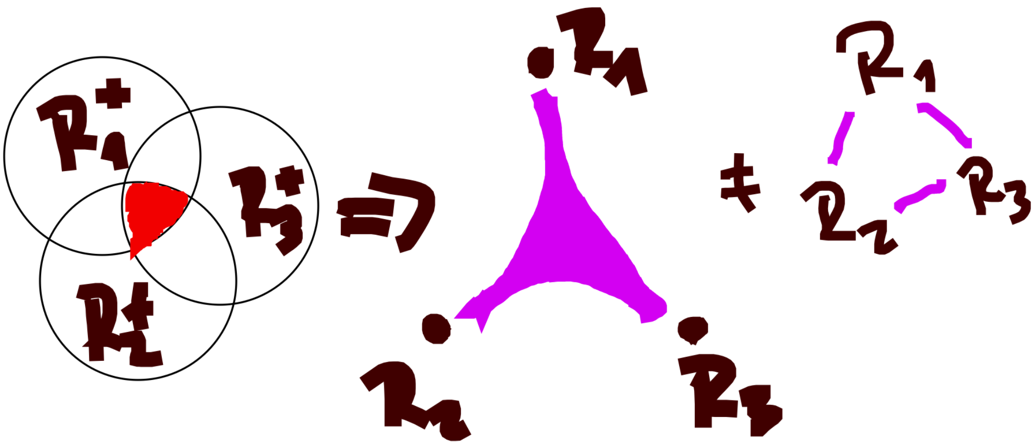
In[ ]:=
```
ms =
  MultiwaySystem[{{{1, 2}, {2, 3}} → {{1, 2}, {2, 3}, {3, 4}}}, {{{1, 2}, {2, 3}}}];
ms["CausalBranchialGraph", 3]
```

Out[ ]=



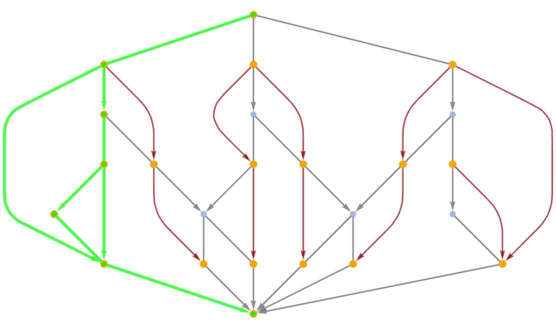
- This is a dynamical description of the multiway evolution in terms of **causality** and **branching**. In the appropriate limit, the resulting geometry should converge to a structure capable of supporting **quantum gravity**.
- Note that there are *higher-order irreducible relations* to extract:

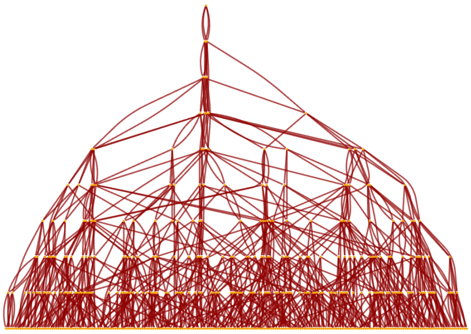




Infrageometry of causal graphs

- Picking a particular updating order—one singular history of the universe—selects a subgraph of the full evolution causal graph. The resulting structure is the (single-way) **causal graph**, whose emergent geometry should support **special relativity**.

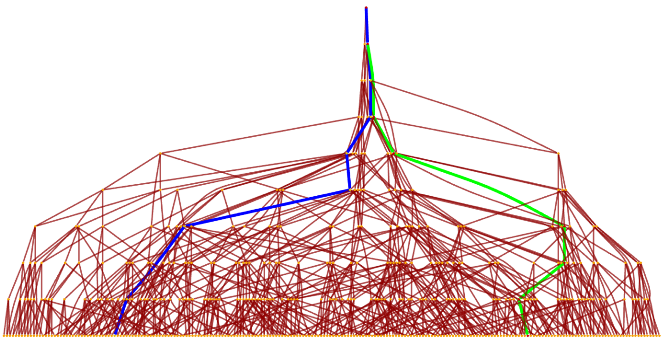




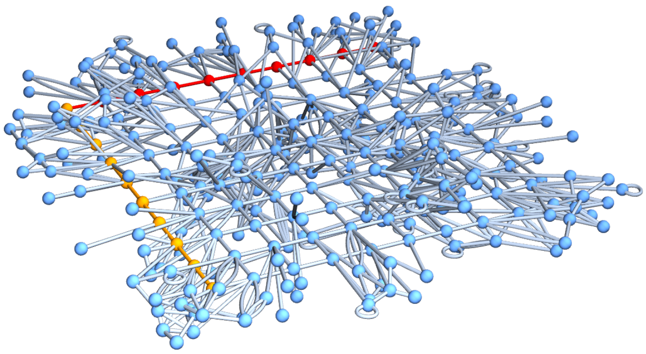
- An important property of the multiway system is **causal invariance**, which requires that the causal graphs for each branch (from a given set) are isomorphic. This guarantees that observers from this set **agree on the same history**. In comparison, confluence guarantees that all branches eventually lead to the same universe.

Infrageometry on spatially reconstructed hypergraphs

- In a causal graph, we pick **observer chains** and assign **coordinates** to events using the shortest-path projection. After transforming from light-cone gauge, we obtain **proper time** and **simultaneity slices**. If we then connect events in a simultaneity slice whenever they have an immediate common ancestor event in the previous slice, we obtain the so-called **spatially reconstructed hypergraph**. This construction is based on the following postulates:
  - minimal model of an observer is a causal edge
  - minimal model of spatial closeness is the existence of an immediate common ancestor



- Multiple events can share the same spatial coordinate, which we interpret as internal degrees of freedom, leading to **graph fibrations**. Because the shortest-path projection is non-unique, we actually deal with **multi-fibrations** in which a single fiber element can lie over multiple base points.

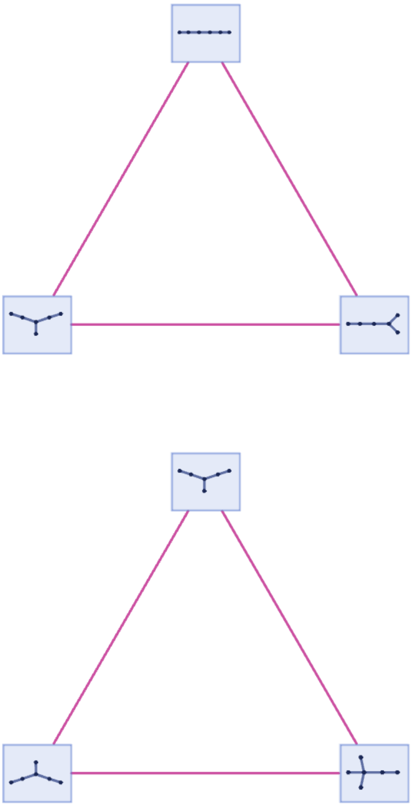


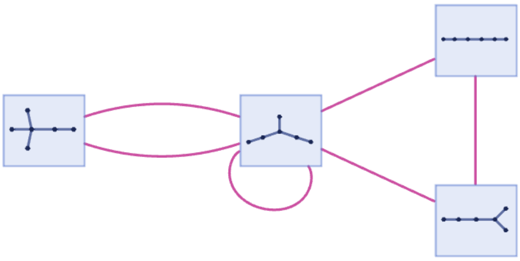
- On the fibrations we build quantum field theory by investigating the moduli space of **minimal models of connections** (bi-partite pairings of fibers), defining integration weights using the **time evolution operator (**bi-partite causal graph between simultaneity slice), and investigating quantum corrections to **Wilson loops** and other observables.

Infrageometry of branchial graphs

- Branchial graphs are constructed from the states graph by connecting states that share an immediate common ancestor. We interpret nodes as **pure quantum states** and connected components as their **superpositions**. This relies on the idea that **branching is the minimal model of non-determinism**.

In[ ]:=
```
Row @ {
  ms["BranchialGraph", 3],
  Spacer[60],
  ms["BranchialGraph", 3, "CanonicalStateFunction" → "CanonicalHypergraph"]}
```
Out[ ]=

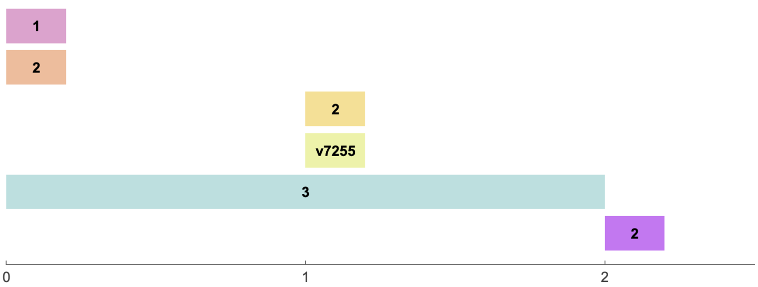




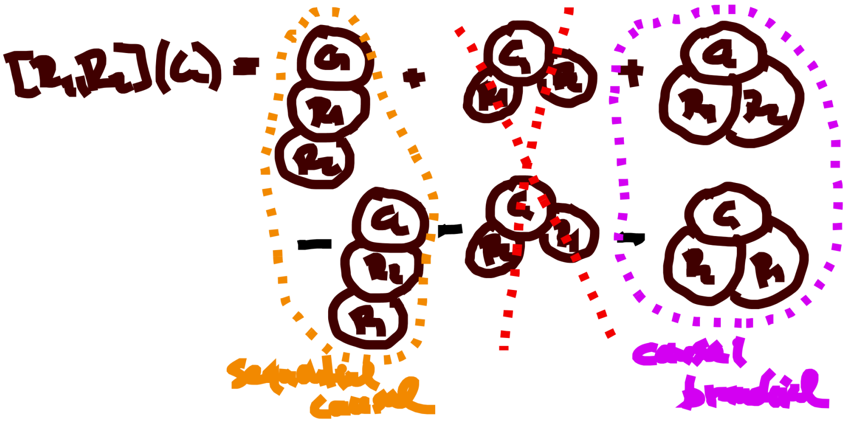
- The resulting geometry should correspond to that of an **operator algebra** together with a **Hilbert space** which is its unitary irreducible representation.

Alternative views

Causal graphs provide a **dynamical perspective** on multiway evolution. There is also a complementary **persistence approach** that studies how **EMEs**—elementary objects with an identity such as particular vertices, edges, or isomorphic subgraphs—propagate throughout the evolution. Here is a **barcode** of vertex emes:

- 

Each rewriting rule corresponds to an **infinitesimal generator of the multiway evolution** on the state spaces. In certain cases, the commutator is a linear combination of other rewriting rules. This occurs for **elementary blow-ups**, meaning rules whose left-hand side consists of either a single vertex or a single hyperedge. The resulting Lie bracket generalizes the classical **insertion Lie bracket** on rooted trees, induces a Poisson bracket on the space of functions on rooted hypergraphs, and may play a role in a renormalization theory for hypergraph-rewriting–based models.



- Example of integration of insertion Lie bracket on vertices up to third order:

$$e^{tx} y e^{-tx} = y + t[x,y] + \frac{t^2}{2!}[x,[x,y]] + \frac{t^3}{3!}[x,[x,[x,y]]] + \ldots$$

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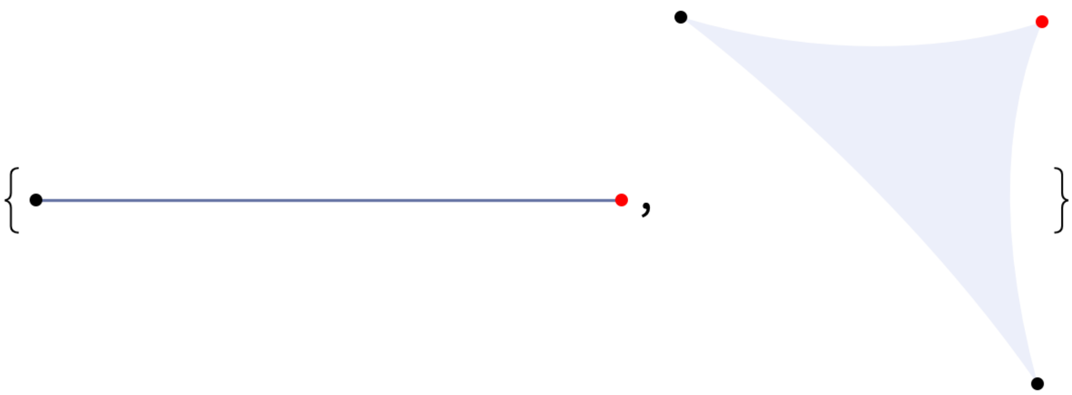

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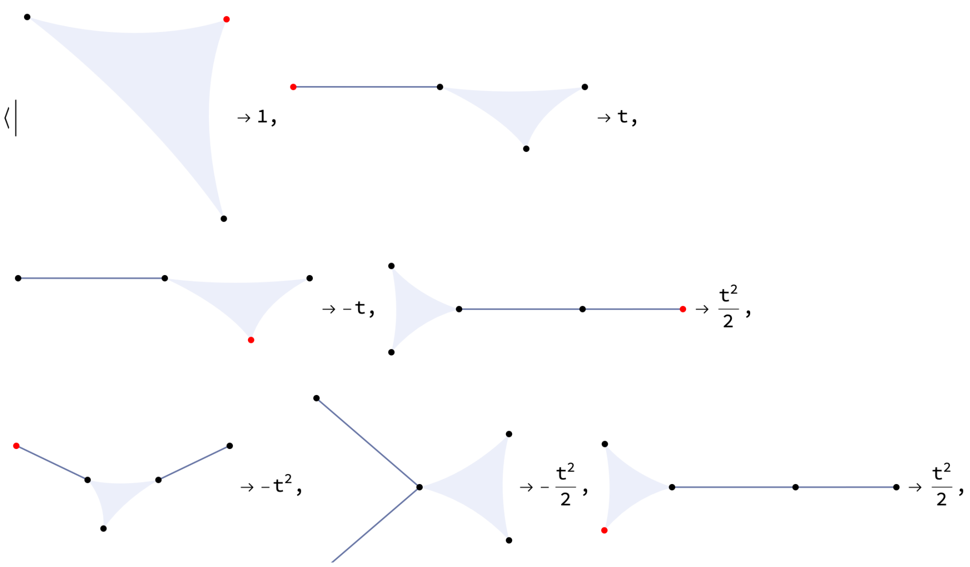
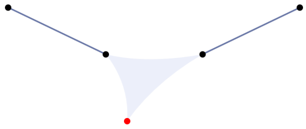
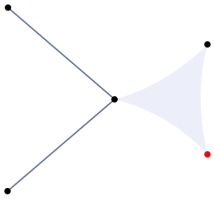
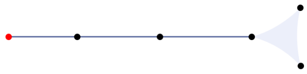
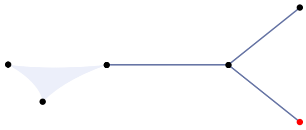
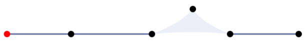
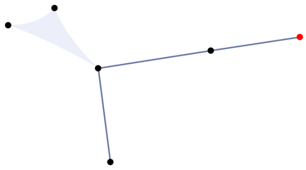
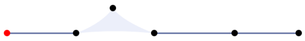
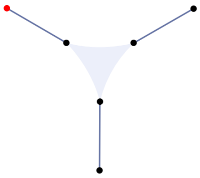
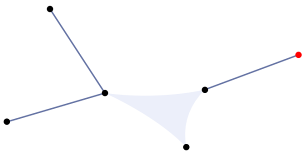
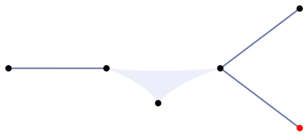
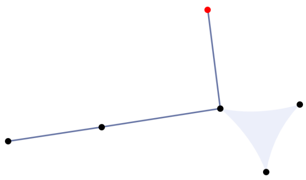
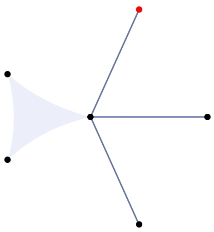
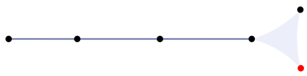


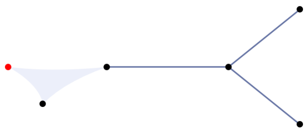
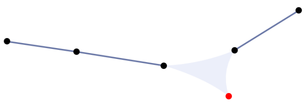
 $\to \frac{t^2}{2}$,  $\to \frac{t^2}{2}$,  $\to \frac{t^3}{6}$,

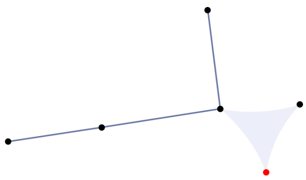
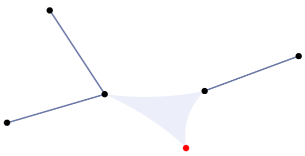
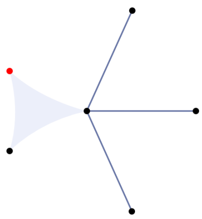
 $\to -\frac{t^3}{6}$,  $\to -\frac{t^3}{2}$,

 $\to -\frac{t^3}{3}$,  $\to \frac{t^3}{2}$,

 $\to \frac{t^3}{2}$,  $\to \frac{t^3}{2}$,  $\to \frac{t^3}{2}$,

 $\to \frac{t^3}{6}$,  $\to \frac{t^3}{6}$,  $\to -\frac{t^3}{6}$,

 $\to -\frac{t^3}{6}$,  $\to -\frac{t^3}{3}$,

 $\to -\frac{t^3}{3}$,  $\to -\frac{t^3}{3}$,  $\to -\frac{t^3}{6} \big|\rangle$

Example 1 : Topology

- An **edge is a minimal model of closeness**, which is justified by:
  - **Theorem of J. Latschev:** For every closed Riemannian manifold M, there exists $\varepsilon > 0$ such that for any $0 < r < \varepsilon$ there exists $\delta > 0$ such that for every metric space X with $d_{GH}(M, X) < \delta$ the **Vietoris–Rips complex** of X at scale r is homotopy equivalent to M.
- Given a **hypergraph**, we take the Vietoris-Rips complex of its 1-skeleton and compute the **barcode** for scales $r = 1, 2, \ldots, \text{diam}(G)$.

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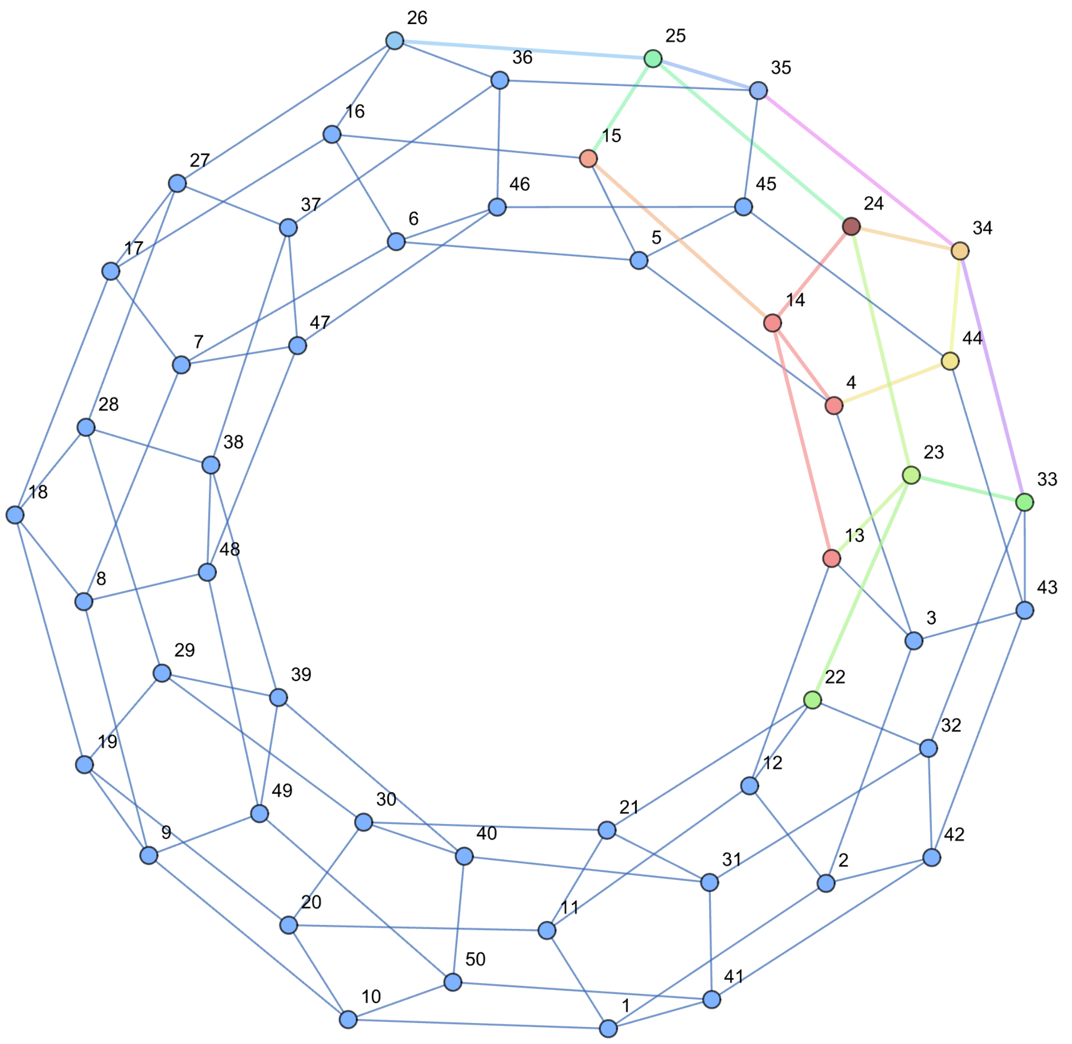

- However, the Vietoris–Rips complex suffers from **combinatorial explosion**, so one must truncate high-dimensional simplices.

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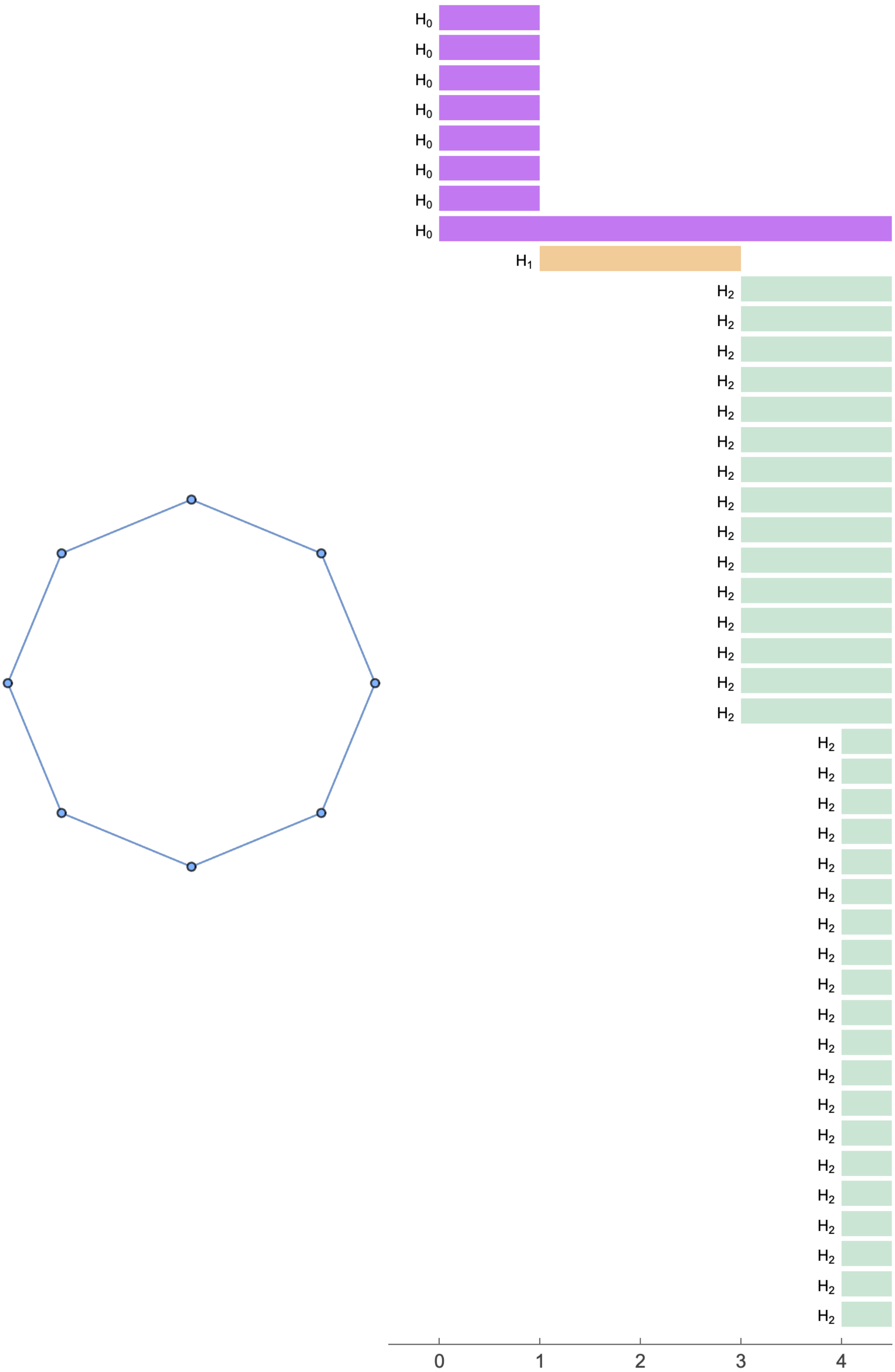

- The following **scale-dependence phenomenon** is exhibited by discrete invariants:
  - **quantum scale** $d = 0$ → the space is fully discretized.

- **mesoscale** $d = 1, 2$ → the homology matches our geometric intuition.
- **coarse scale** $d > 3$ → the space collapses to a point.

  (The appearance of higher-dimensional homology generators is an artifact of truncating the Vietoris–Rips complex to avoid combinatorial explosion.)

- The Vietoris–Rips complex is a **finite ACS**, so it admits a **canonical Hodge decomposition.** One can therefore use the associated invariants, such as the **spectrum of the Hodge Laplacian** (whose vertex and edge components are the Gram matrices of the incidence matrix) to say more about the intrinsic geometry.
- One can even fix some orientation and consider **discrete version of Chern-Simons theory** (IBL-infinity) to extract “higher order information” (loop spaces).
- There even exist notions of **geodesics on ACS** when the simplices in a given dimension satisfy the unique-mirror property (i.e., when the complex is a **Dehn–Sommerfeld q-manifold**). [Oliver Knill]

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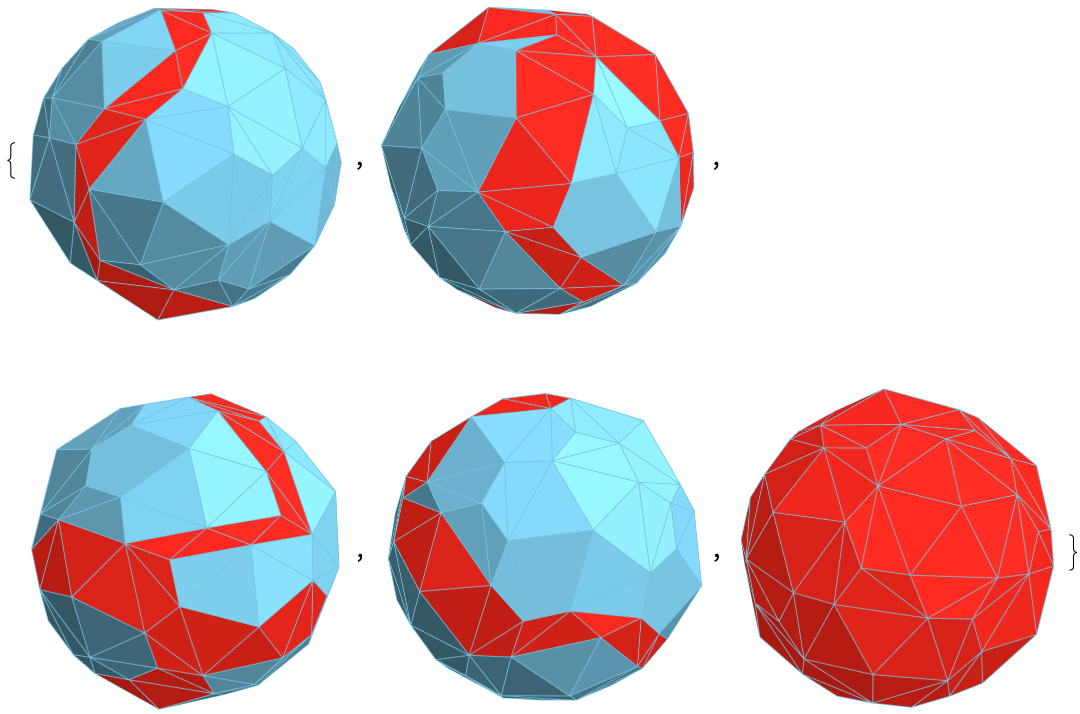

Example 2: Dimension

- Neither the dimension of the “model space” (unless one specifies “model rewriting rules of given dimension”) nor the Hausdorff dimension is useful here: the former is yet undefined, and the latter is either 0 (a discrete set) or 1 (a 1-dimensional CW complex)
- For any **Riemannian manifold:**

$$V(r) = \frac{\pi^{d/2}}{(d/2)!} r^d \left(1 - \frac{r^2}{6(d+2)} R + O(r^4)\right)$$

- **Wolfram-Hausdorff dimension:**

$$\Delta(r) = \frac{d \log(V(r))}{d \log(r)} \sim \frac{\log(V(r+1)) - \log(V(r))}{\log(r+1) - \log(r)}$$

  with respect to the **graph distance.**
- This corresponds to **one edge as minimal model of distance** and is a **first-order metric invariant.**
- Here we visualize the mean Δ for $r \in [r_{\min}, r_{\max}]$ for each point in the selected neighborhood and for the neighborhood as a whole.

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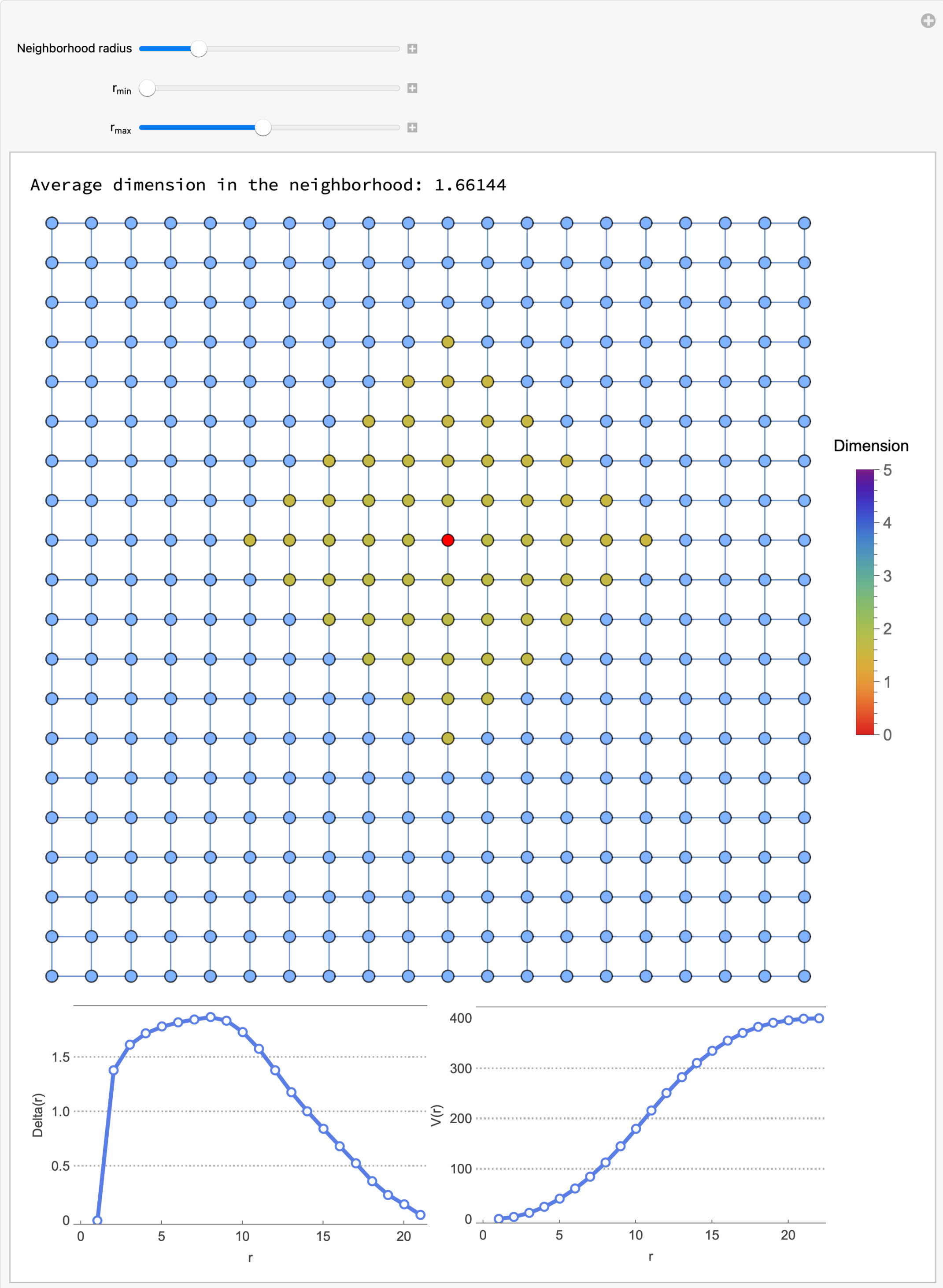

- **Scale-dependence phenomenon**: The quantity $\Delta(r)$ is for large r influenced by the **finiteness of the graph,** and for small r by **local irregularities**. We are thus interested in a **plateau** where $\Delta(r)$ is approximately constant.

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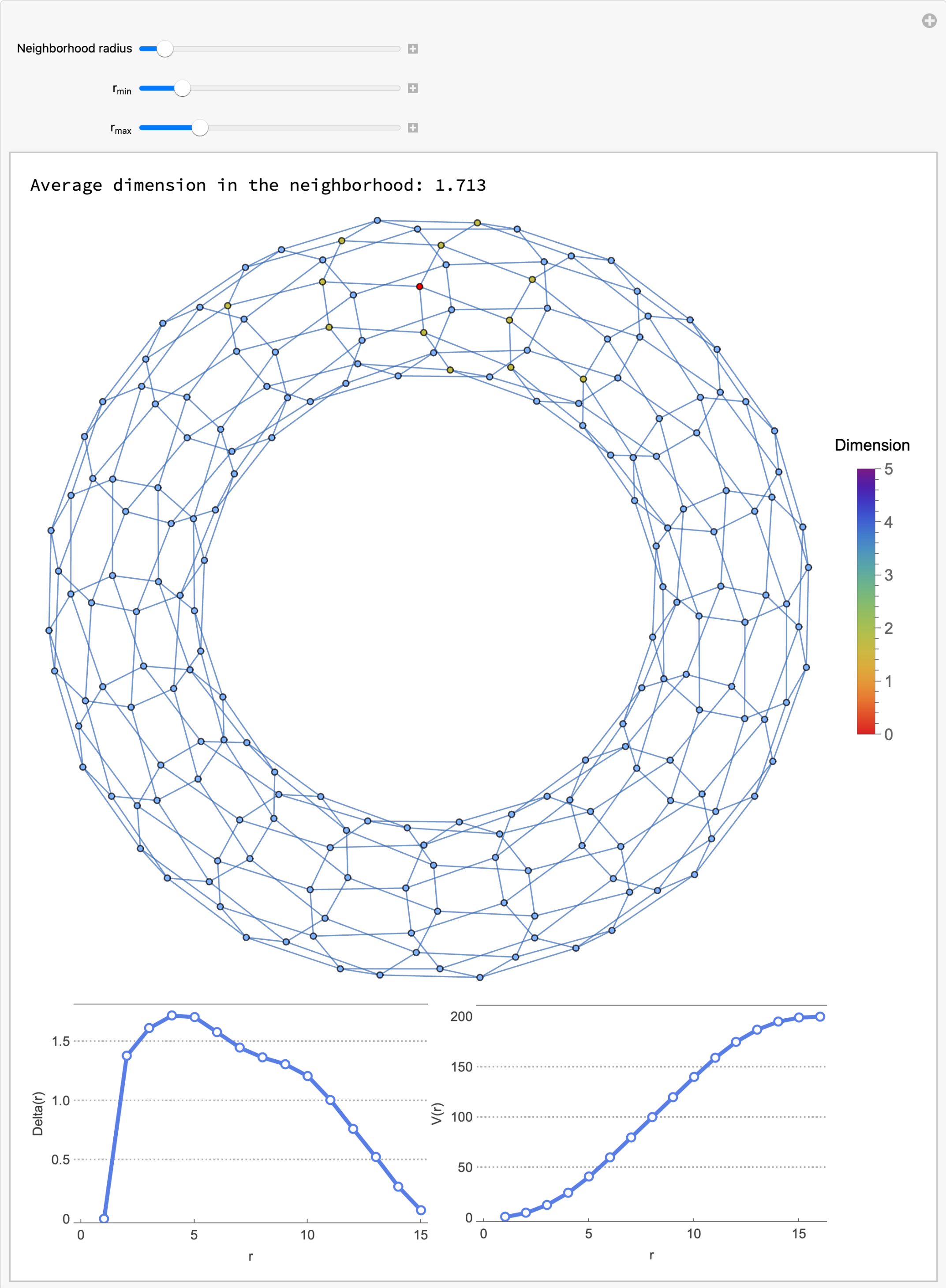

- There could be other notions of dimension (e.g. based on homology, coordinatization, causal graph, model rules), and their equality can be used as a **test admissible initial conditions**. There are also other graph metrics, e.g., **effective resistance**.

Example 3 : Straight line

- **Higher-order metric notion** based on **shortest distance as a minimal model of straightness.**
- The space of shortest paths is largely degenerate. We can try to select those lying in the **center** with respect to the **Fréchet distance** (having the least metric eccentricity).

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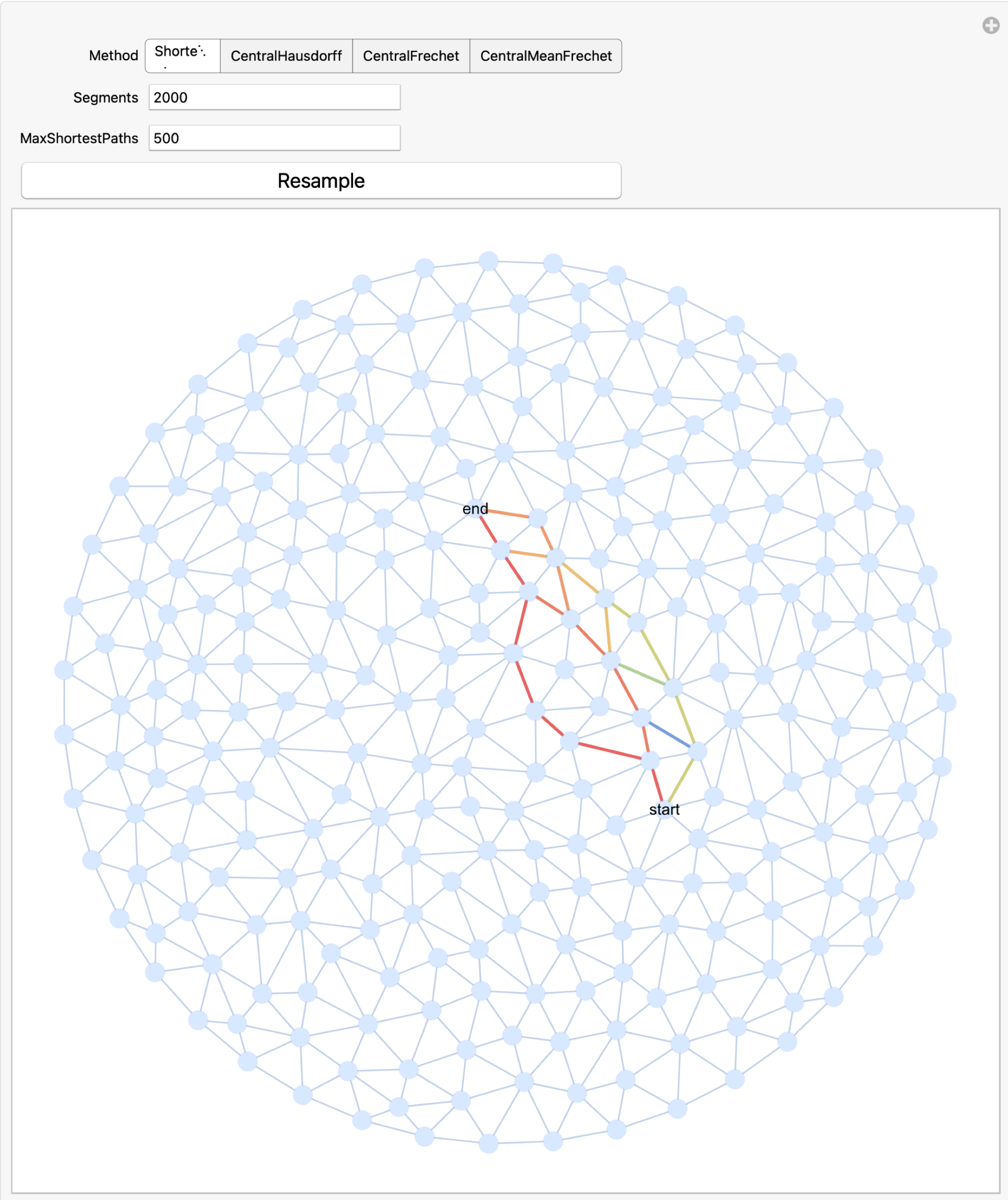

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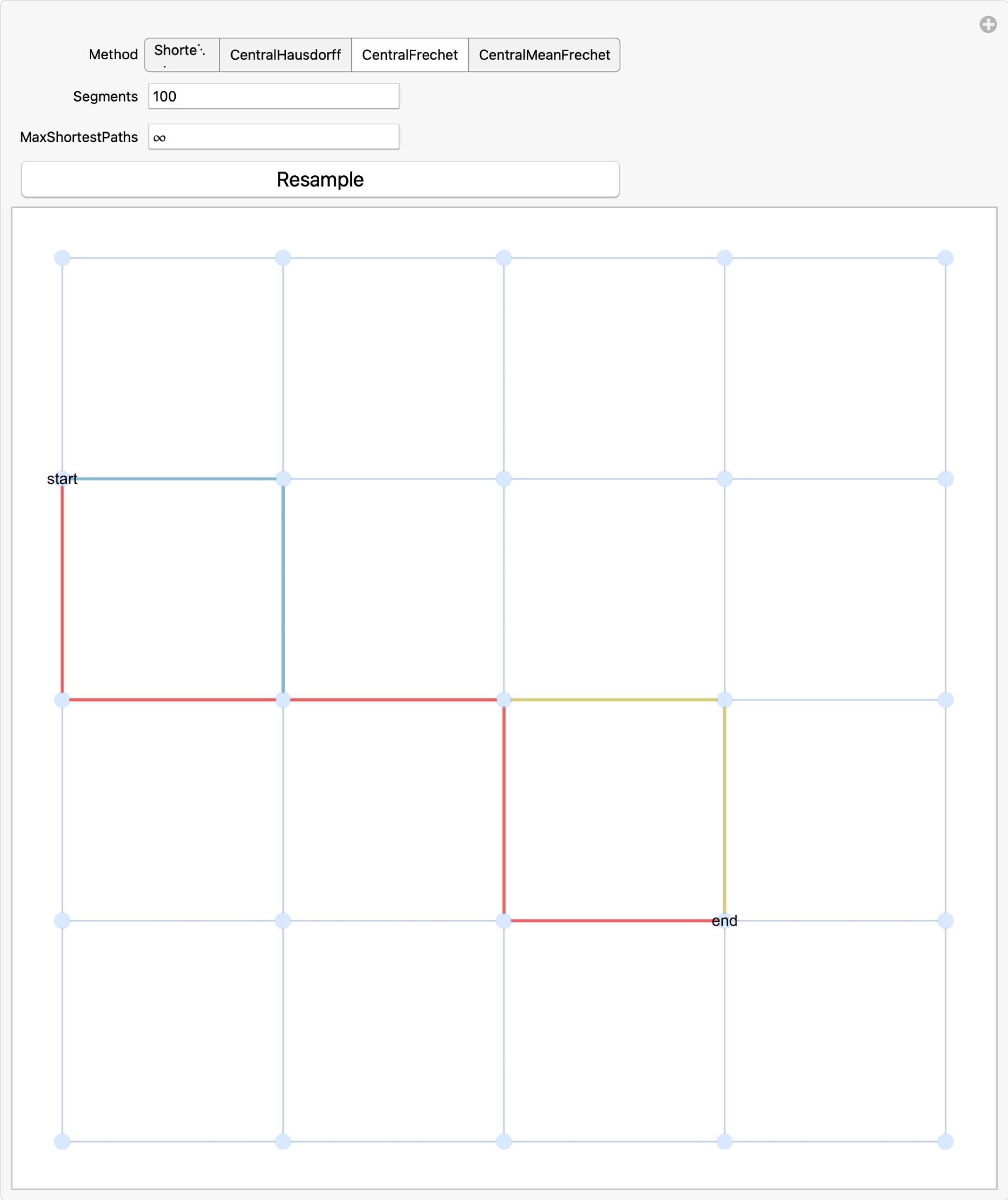

- We try to define analogous notions of sphere, plane, circle, and cylinder, and to develop a **synthetic infrageometry**, followed later by an **axiomatic infrageometry**. This will involve studying the collective behavior of the results **multi-constructions.**

Example 5: Simultaneity and Light rays

- An edge is a **minimal model of causality**.
- A **foliation**—namely, a decomposition into successive antichains (an oriented fibration over the natural numbers)—corresponds to the **simultaneity slices** (Cauchy surfaces) associated with an observer.

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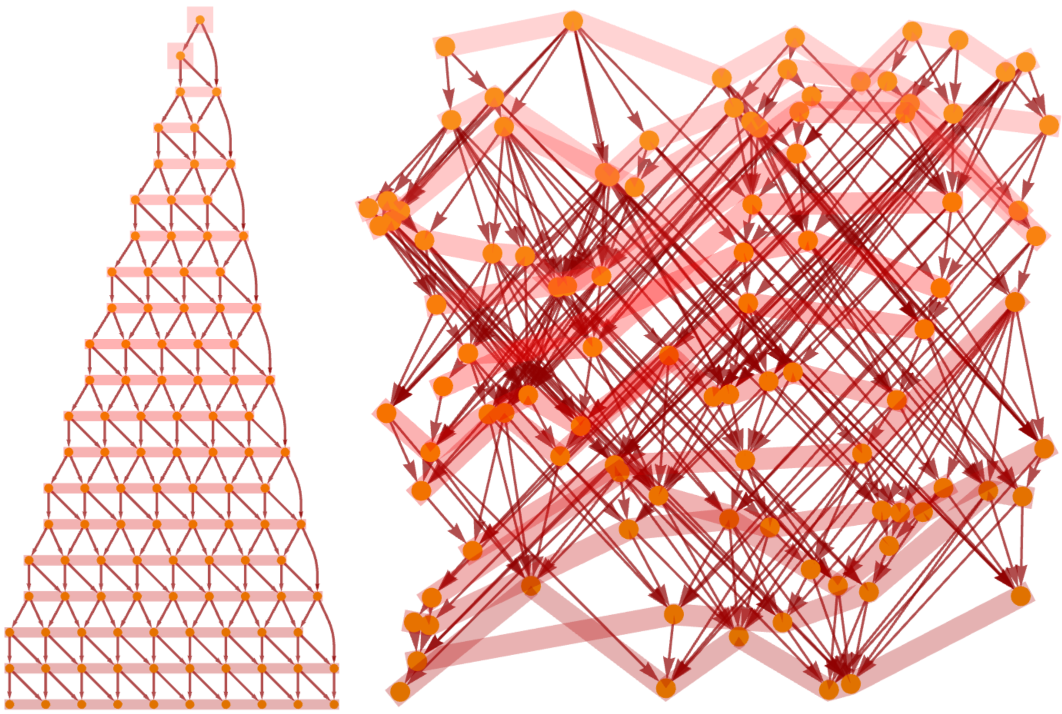

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- Light rays are **observer-independent**, and a light ray from a point is defined as the subgraph whose **causal interval** between each vertex and the origin of the light cone is **degenerate**.

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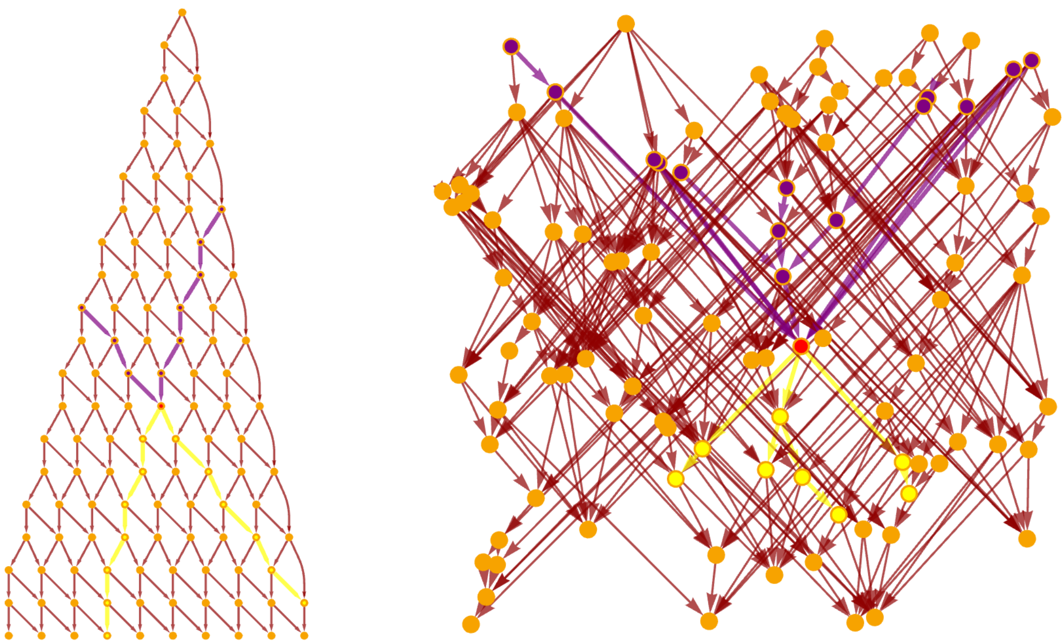

- We also attempt to define **event horizon**, **inertial frames**, study **Lorentz transformation**, or define **mass**.
- We attempt to study axiomatic special relativity and to determine which graph substrates can support it.

Example 6 : Fibered Graph

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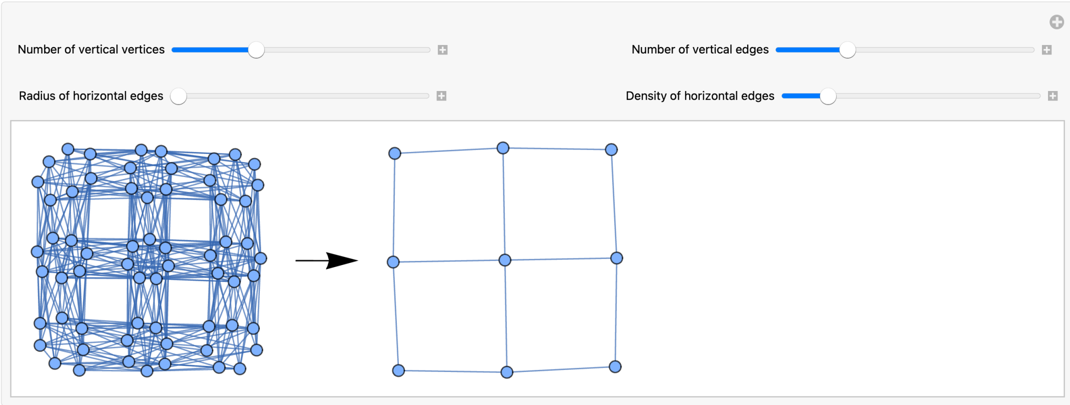

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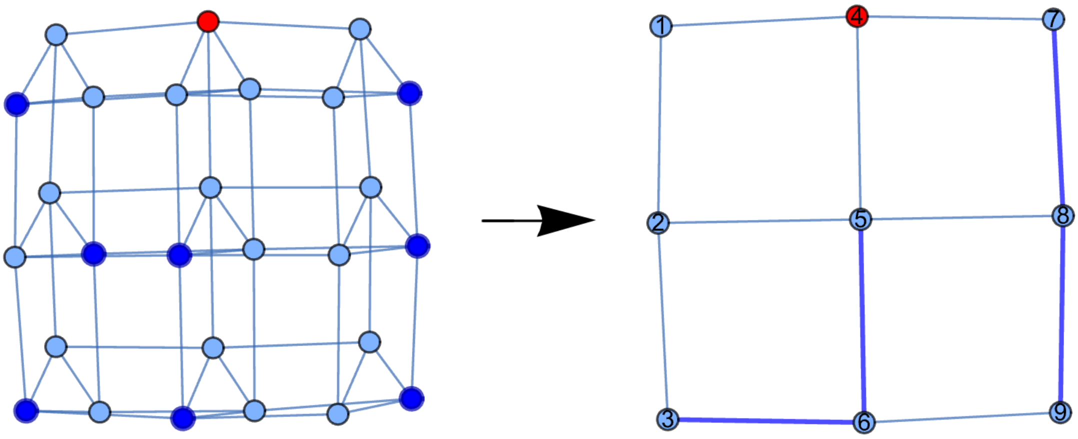

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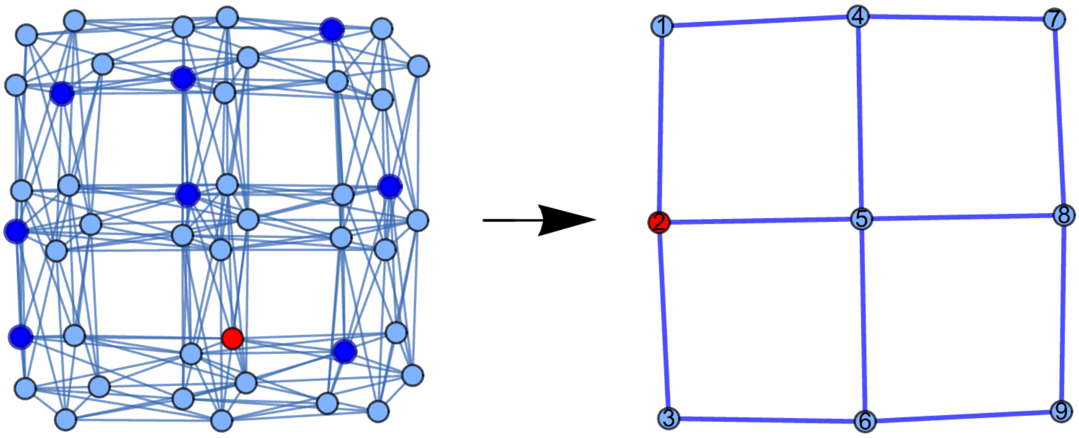

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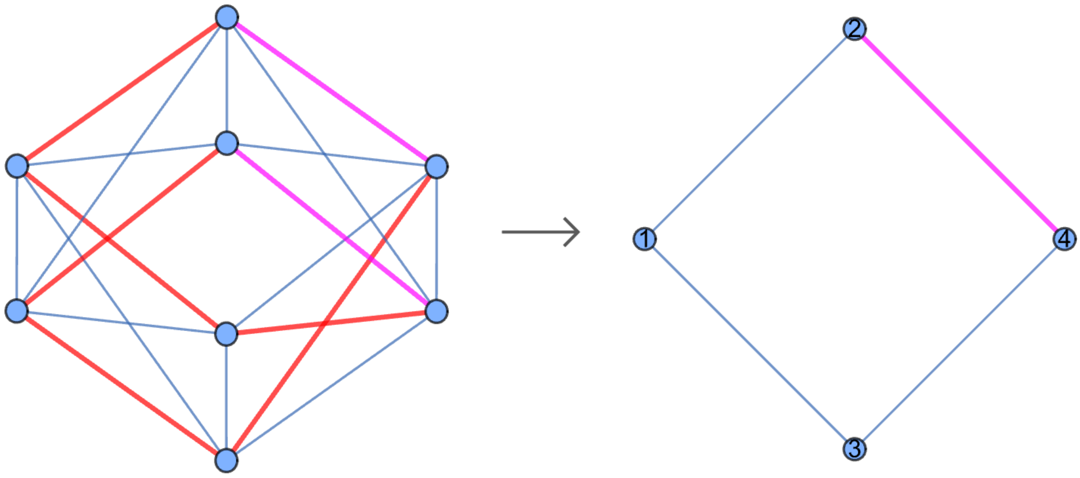

- **Minimal model of fiber bundle with chosen gauge and with a connection.**
- Having a smooth section, we can embed it in the plane and anchor the fibers to it.

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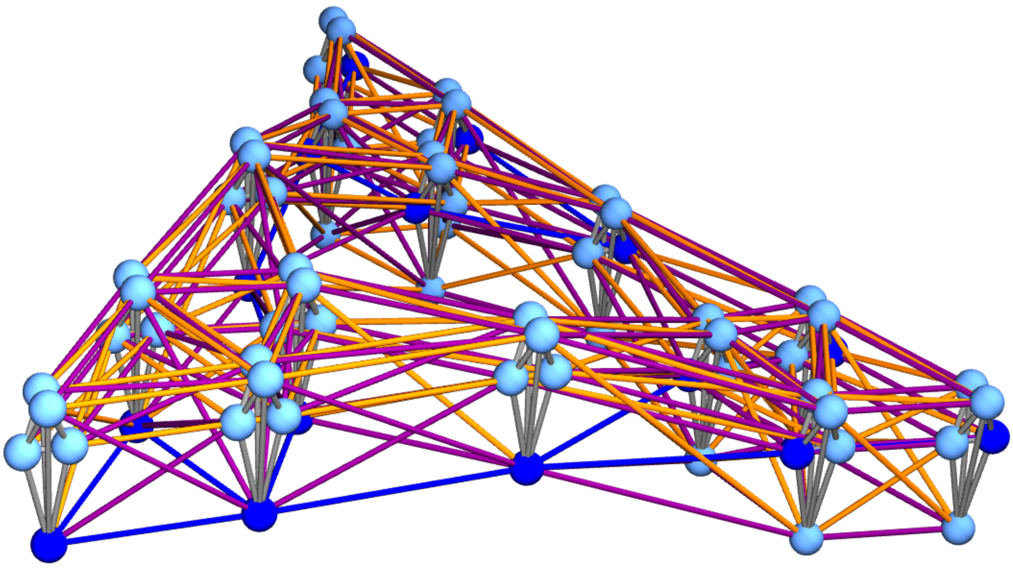

- Now we can do **horizontal lifting** and compute **holonomy** and its trace = **Wilson loop**.

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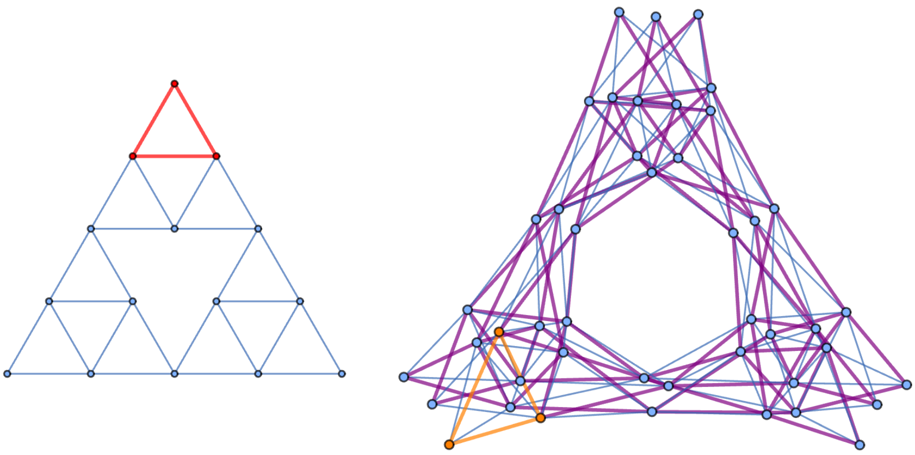

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Example 6: Tangent Bundle and Exponential Map

- The **tangent bundle** is a fibered graph with fibers tangent vectors and horizontal edges being edges of maximal transverse bipartite graph matchings.

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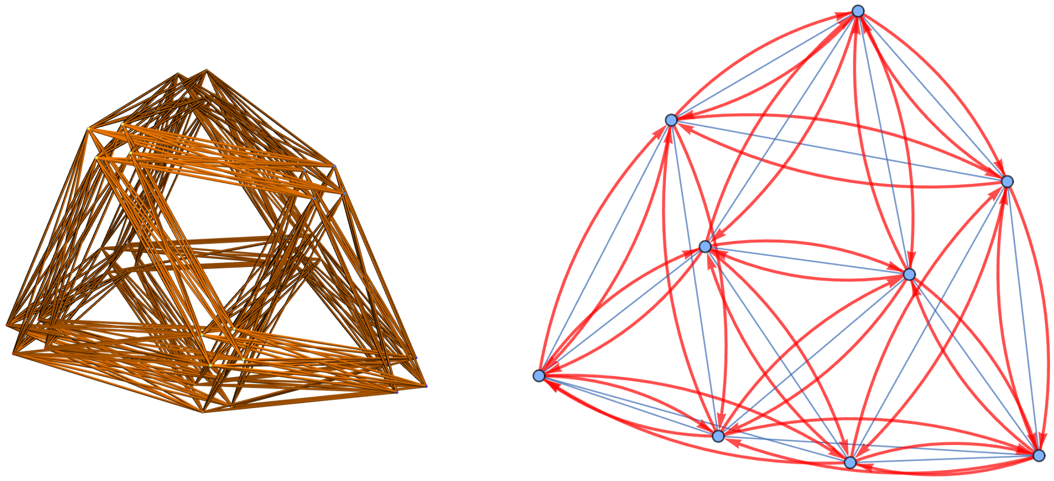

- The notion of a **smooth section** is a $d = 1$ notion, since it considers only immediate neighbors. However, we already have a $d = 2$ notion of smoothing of vector fields. We should define smoothening of sections for each d.

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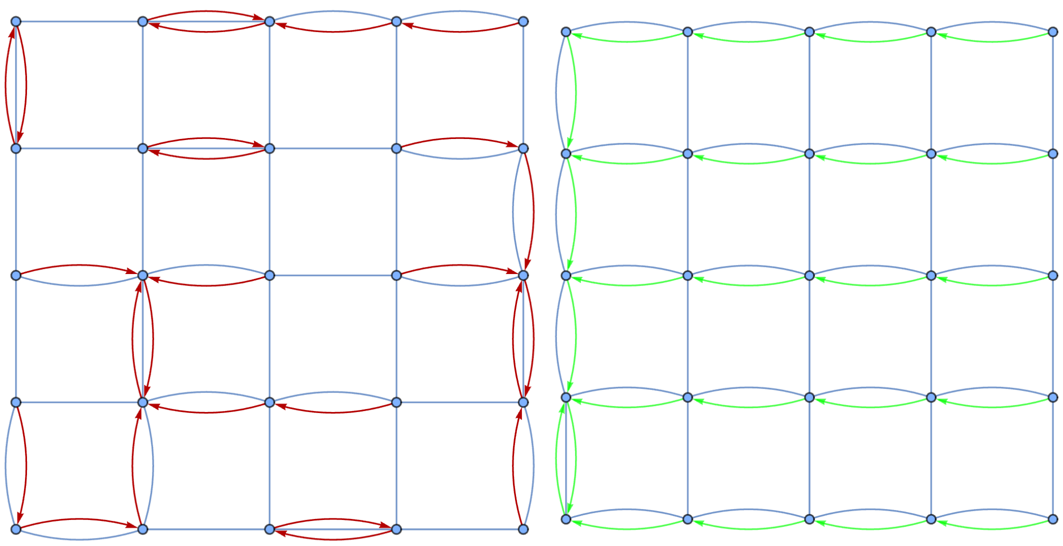

- **Riemannian exponential map**

  $\exp_p : T_p M \to M$ sends $v \in T_p M$ to $\gamma(1) \in M$

  reached by the unique geodesic $\gamma : [0, 1] \to M$ with $\gamma(0) = p$ and $\gamma'(0) = v$.

A proposal for minimal model :

For every $x \in G$ and $v_x \in T_x G$ let $n(v_x, y)$ be the number of shortest path from x to y starting with edge v_x .The exponential coordinates of y with respect to the basepoint x is defined as the linear combination $\frac{\sum_{v_x \in T_x G} n(v_x, y) v}{\sqrt{\sum_{v_x \in T_x G} n(v_x, y)^2}}$.

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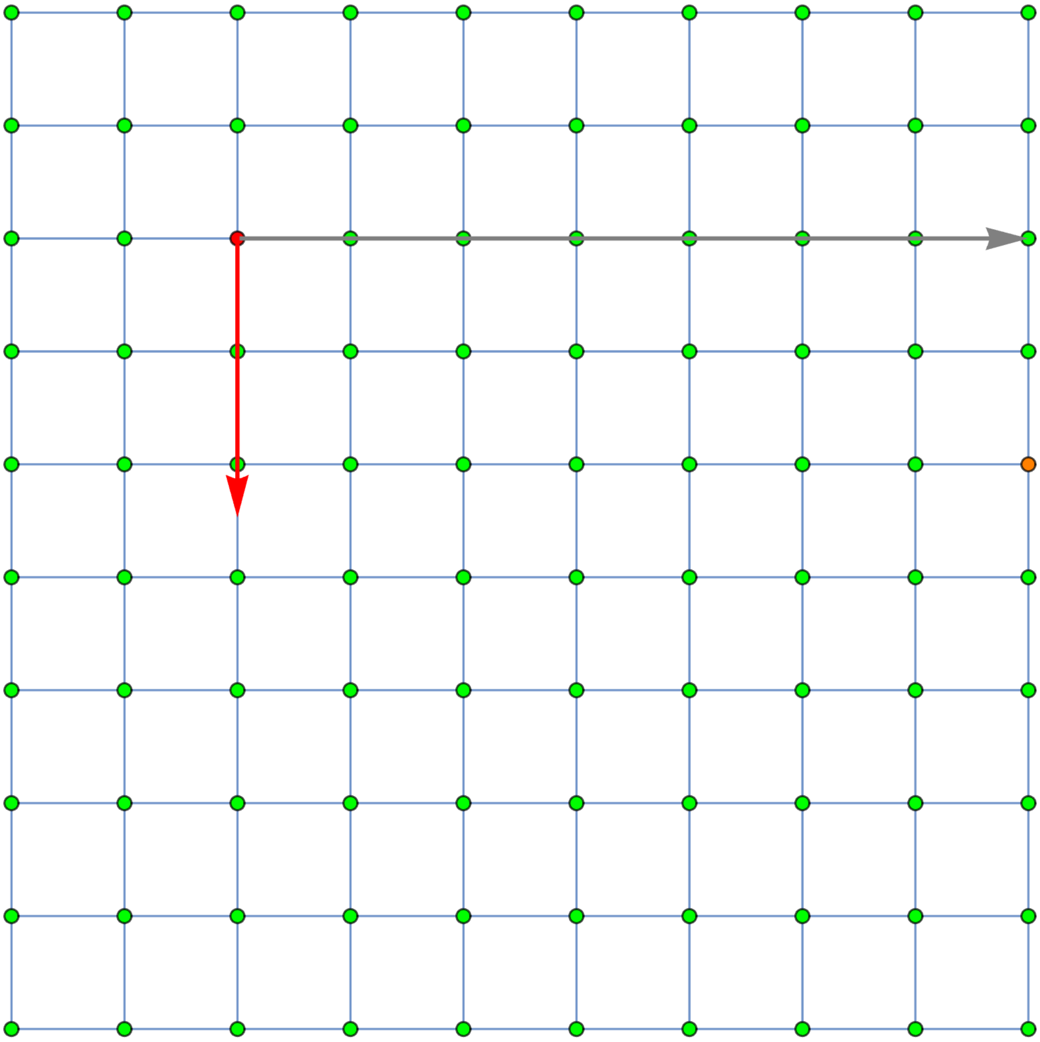

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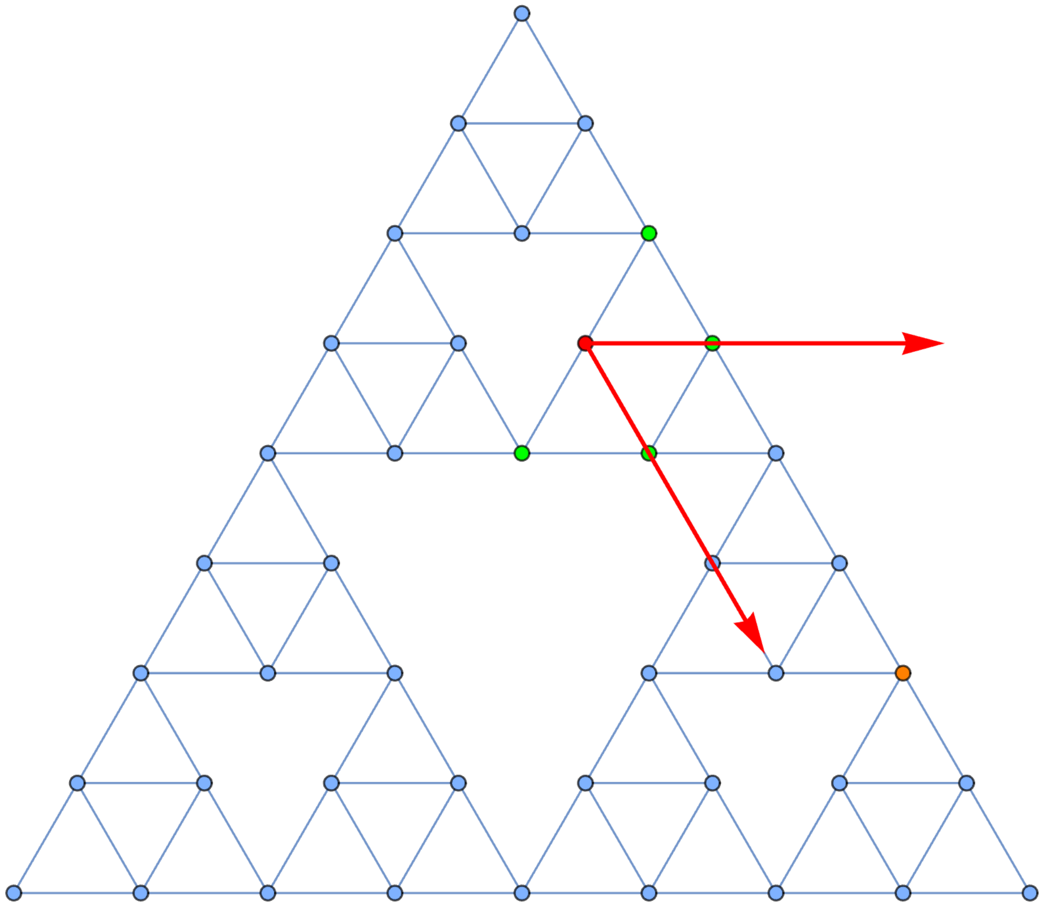

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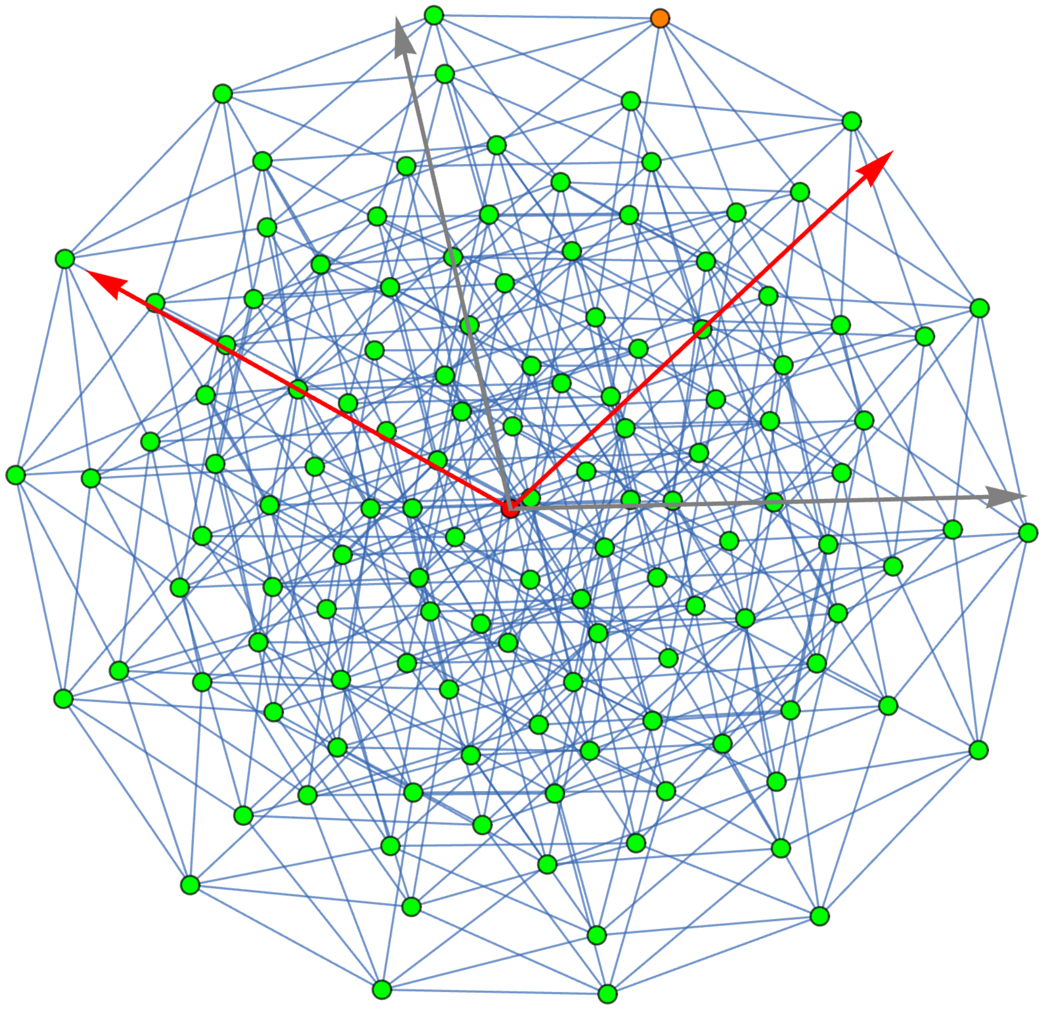

- -v ?
- Lie bracket [X, Y]? Affine connection?
- Vector space as limit under subdivision?