On Manifolds with Corners

Pavel Hájek

June 5, 2014

Motivation

- Prof. Dr. Kai Cieliebak, **Lectures on String Topology**; Universität Augsburg, WS 12/13. (Lecture Notes Available Online)

- To understand the details of Chas & Sullivan's construction of the **Loop Product**; String Topology (arXiv:math/9911159)

- **Geometric Homology** : Replace standard simplexes by manifolds with corners such that intersections can be considered.

Geometric Homology

- **Geometric Simplex** $[P, f]$: P manifold, $f : P \to X$ continuous
- **Boundary** $\partial[P.f] = \sum_i [\partial_i P, \partial_i f], \quad \partial P = \bigcup_i \partial_i P$
- **Geometric Chain Complex** (P, ∂) with relations:

$$[-P, f] = -[P.f]$$

$[P, f] = 0$ if $[P, f]$ is degenerate

$[P, f] = [P_1, f\big|_{P_1}] + \ldots + [P_n, f\big|_{P_n}]$ if $P = \bigcup_{i=1}^n P_i$

Tasks

- Carry out the construction in details.
- Prove $\mathcal{HP}(X) \simeq H(X)$.
- Define the loop product on $\mathcal{HP}(\mathbb{L}M)$.

What has been done...

- Differential topology of manifolds with corners:

  neat immersions / submersions

  manifolds with embedded faces

  transverse intersection and approximation

  vector fields and collar neighborhoods

  basic properties of decompositions

  orientations

- Plan how to prove the tasks using the facts above + two conjectures about triangulations. (Not in the thesis)

Motivation

Manifolds with embedded faces = Manifolds with Corners such that the 1-faces are embedded manifolds with corners (and automatically manifolds with embedded faces)

Vector Fields Compatible with Boundary I

Vector Fields Compatible with Boundary I

- In a manifold with corners the *flow of a vector field* does not always exist.

Vector Fields Compatible with Boundary I

- In a manifold with corners the *flow of a vector field* does not always exist.

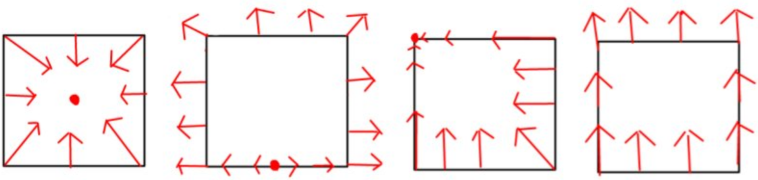
No integral curve exists at marked points in $\mathbb{H}^2$ and $\mathbb{R}^2_+$, respectively.

Vector Fields Compatible with Boundary II

- Restrict to the class of *vector fields compatible with boundary*: The vector field X is either inward pointing or outward pointing or tangent to F at all points of F for any 1-face F.

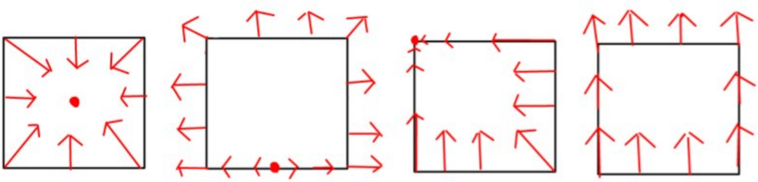
Vector Fields Compatible with Boundary II

- Restrict to the class of *vector fields compatible with boundary*: The vector field X is either inward pointing or outward pointing or tangent to F at all points of F for any 1-face F.



Vector Fields Compatible with Boundary II

- Restrict to the class of *vector fields compatible with boundary*: The vector field X is either inward pointing or outward pointing or tangent to F at all points of F for any 1-face F.



- Then the flow $\psi : W \to P$ of X exists where $W \subset P \times \mathbb{R}$ is its not necessarily open domain.

Collar Neighborhoods I

Recall: *Collar neighborhood* of a 1-face F of a manifold with embedded faces P is an open neighborhood $C(F)$ of F together with a diffeomorphism $\varphi_F : F \times [0,1) \to C(F)$.

Collar Neighborhoods I

Recall: *Collar neighborhood* of a 1-face F of a manifold with embedded faces P is an open neighborhood $C(F)$ of F together with a diffeomorphism $\varphi_F : F \times [0,1) \to C(F)$.

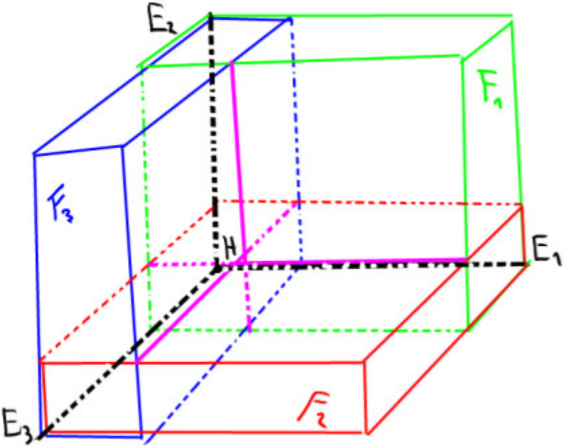
Theorem (Collar Neighborhood Theorem)

Every 1-face F of a compact manifold with embedded faces P has a collar neighborhood $C(F)$ such that the diffeomorphisms φ_i for different F_i commute on overlaps of domains and

$$C(F_1) \cap \ldots \cap C(F_n) = \bigsqcup_{\substack{H \text{ } n\text{-face of } P \\ H \subset F_1 \cap \ldots \cap F_n}} C(H)$$

where $C(H)$ is a neighborhood of H diffeomorphic to $H \times [0,1)^n$ through $\varphi_1 \circ \ldots \circ \varphi_n$.

Collar Neighborhoods II



Conclusion

- The basic framework and tools to work with manifolds with corners are established.

Conclusion

- The basic framework and tools to work with manifolds with corners are established.

Remains to be proven:

Conclusion

- The basic framework and tools to work with manifolds with corners are established.

Remains to be proven:

- Triangulations T_i of some 1-faces F_i which agree on overlaps $F_i \cap F_j$ can be extended to a triangulation T of P.

Conclusion

- The basic framework and tools to work with manifolds with corners are established.

Remains to be proven:

- Triangulations T_i of some 1-faces F_i which agree on overlaps $F_i \cap F_j$ can be extended to a triangulation T of P.
- Given two triangulations T_1, T_2 of P there is a third triangulation T_3 and a diffeomorphism $\psi : P \to P$ such that T_3 is finer than T_1 and $\psi(T_2)$.

Thank you for your attention !