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On Manifolds with Corners

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Selbstständigkeitserklärung

Ich erkläre hiermit, dass ich die vorliegende Arbeit selbstständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel angefertigt habe.

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Abstract: The aim of this thesis is to explore properties of manifolds with corners. The main motivation is to use them to define a geometric homology theory which would carry an intersection product induced from intersections of transverse chains and which would agree with the singular homology theory. In this thesis, manifolds with embedded faces are defined which we believe are the right objects to generalize simplexes. Several results about their differential topology are proven; in particular, results about transverse intersections and approximations, orientations, neat embeddings, and compatible collar neighborhoods. Most importantly, a self-consistent framework to work with manifolds with corners and tackle related problems is set up.

Keywords: manifolds with corners, transversality, orientation, collar neighborhoods, decompositions, geometric homology

1 Introduction

The main motivation for studying manifolds with corners was a lecture String Topology given by Prof. Dr. Kai Cieliebak in Winter Term 2012 at University of Augsburg. Lecture notes are available online [1]. One of the highlights was the definition of the loop product and related structures on the loop homology of a manifold. Before we discuss the thesis, let us recall the construction of the loop product in the following paragraph. In fact, my ultimate goal is to understand this construction rigorously. The presented master's thesis may be seen as a step towards this goal.

String topology was developed by Chas and Sullivan in their pioneering article "String Topology" [2] pre-printed on ArXiv. They consider families of loops $x : K_x \times \mathbb{S}^1 \to M$ and $y : K_y \times \mathbb{S}^1 \to M$ in a connected manifold M. The idea is to concatenate them in a particular way so that one gets a third family of loops

$$x \bullet y : K_{x \bullet y} \times \mathbb{S}^1 \to M.$$

It is done by letting $K_{x\bullet y}$ to be the set of points $(p, q) \in K_x \times K_y$ such that $x_p(0) = y_q(0)$ and letting $(x \bullet y)_{(p,q)} : \mathbb{S}^1 \to M$ to be the loop which traces x_p first and y_q second at twice the velocity of x_p and y_q, respectively. This construction is called the *loop product* of x and y. The authors use this to define a product on homology of the space of parametrized loops in M which is denoted by $\mathbb{L}M$. The correspondence is obvious: If $K_x = \Delta^n$ is the standard n-simplex then we may consider $x : \Delta^n \to \mathbb{L}M$ defined by sending $p \mapsto x_p$ as a singular n-simplex in $\mathbb{L}M$ and similarly for y if $K_y = \Delta^m$. However, when the loop product of two such singular simplexes is considered, we see that the space $K_{x\bullet y} = \Delta^n \times_M \Delta^m$ is rarely a standard k-simplex; hence, $x \bullet y$ is rarely a singular k-simplex. We may still hope that $x \bullet y$ defines a singular chain, i.e. a collection of singular simplexes. This is true if $K_{x \bullet y}$ can be triangulated. However, not every space has to be triangulable and even if it is, there is no natural way to triangulate canonically. Therefore, in order to get a well defined product on homology one has to check that chains which arise from different triangulations are homologous. The authors do not discuss these technical details. They define a *cell* to be a family of loops x such that K_x is a manifold and consider $x \bullet y$ just for those cells x and y for which the maps $x^0 : p \in K_x \mapsto x(p, 0) \in M$ and $y^0 : q \in K_y \mapsto y(q, 0) \in M$ are transverse. In this case, $K_{x \bullet y} = K_x \times_M K_y$ is a manifold again and $x \bullet y$ is indeed a new cell. In this sense, the loop product is *transversally defined* on chains. Then it is argued that every two chains can be homotoped within their homology class to a pair of appropriately transverse chains. The loop product hereby extends to the homology.

We have seen how the loop product arises in a nice geometric way; however, with several technical details left out. Mathematicians Ralph L. Cohen and John D. S. Jones published a rigorous construction of the loop product for closed manifolds based on standard theorems in their article "A homotopy theoretic realization of string topology" [3]. It shows the loop product indeed exists at least for closed manifolds. Nevertheless, this construction is not geometrically transparent and not accessible without advanced knowledge of homotopy theory.

The idea presented by Prof. Cieliebak in his lecture was to take cells whose domains are manifolds with corners and construct a generalized homology theory for them. In other words, a homology theory where the role of standard simplexes is replaced by a more general class of objects. Such a generalized homology theory will be called a *geometric homology theory* throughout this text. With geometric homology in hands we would be able to look at the Chas-Sullivan's construction and prove it step by step. Eventually, we would prove that the geometric homology agrees with the singular homology. Although the proofs were outlined in the lecture, there are still details left. The aim of this thesis is to work out some of them.

String topology has rich applications in theoretical physics; in particular, in string field theory. At the end of this introduction, let me just briefly mention which algebraic structure with physical relevance arises in string topology: If $\mathcal{H}_i$ is the i-th homology group of $\mathbb{L}M$ then we set $\mathbb{H}_i = \mathcal{H}_{i+d}$. We also set $\mathbb{H}_* = \mathbb{H}_{-d} \oplus \mathbb{H}_{-d+1} \oplus \ldots \oplus \mathbb{H}_0 \oplus \mathbb{H}_1 \oplus \ldots$ and call it the *loop homology* of M. In other words, $\mathbb{H}_*$ consists of the same homology groups but the grading is shifted by d. The loop product is then a graded commutative and associative product $\bullet : \mathbb{H}_i \otimes \mathbb{H}_j \to \mathbb{H}_{i+j}$. Furthermore, there is an operator $\Delta : \mathbb{H}_* \to \mathbb{H}_*$ of degree 1 which is defined for a cell x by the prescription $\Delta x : (u, p, t) \in \mathbb{S}^1 \times K_x \times \mathbb{S}^1 \mapsto x(p, u + t)$. It may be shown that $\Delta^2 = 0$ and that $(\mathbb{H}_*, \bullet, \Delta)$ has the structure of a *Batalin-Vilkovisky algebra.*

In physics as well as in mathematics it is important to have a feeling and intuition for objects one works with. In this sense, the thesis might be found useful as well because it prepares a way for justifying the geometric ideas behind string topology.

2 Manifolds with corners

This main chapter is entirely devoted to manifolds with corners. A manifold with corners is a topological space P with a smooth atlas such that every chart is a homeomorphism of an open subspace of P and an open subspace of the Euclidean corner $\mathbb{R}^n_+$. In particular, the standard n-simplex Δ^n is a manifold with corners and manifolds with / without boundary can be realized as manifolds with corners as well.

In Sections 2.1 to 2.8 we will develop differential topology of manifolds with corners for the purpose of geometric homology. In particular, we will prove a theorem about existence of transverse intersection of manifolds with corners in a boundaryless manifold and a theorem which asserts that we can homotope maps so that they become transverse. We will also prove several other theorems important for being able to handle manifolds with corners. In Section 2.9 we will consider orientations which will become important in a future study of algebraic properties of the intersection product on geometric homology. In Section 2.10 we will concern ourselves with collar neighborhoods which we will use to generalize triangulation results to manifolds with corners (not contained in the text). In Section 2.11 we will define decompositions of manifolds corners; in particular, a geometric triangulation.

The ideas and techniques to prove theorems about manifolds with corners are not very innovative but is is a lot of work. Similar ideas apply in the case of manifolds without boundary and can be found in the standard books about smooth manifolds [4], [5] or in books about differential topology [6], [7]. With few exceptions, however, I have not found any reference where the proofs for manifolds with corners are actually done or a consistent theory is constructed. These exceptions are:

(1) A survey article [8] where existence of the fiber product $P \times_R Q$

is proven for all manifolds with corners P, Q, R; however, with different definitions: In order to overcome singular cases when a 1-face of a manifold with corners (see Section 2.2 for the definition) is not a manifold with corners, abstract points are added to make it a manifold with corners again. Of cource, such a 1-face does not injectively map into the original manifold; hence, is not embedded. From this point of view, our definitions are more geometric. I used this article to get a glimpse of manifolds with corners and of the basic definitions such as the depth and strata.

(2) A detailed book [9] with primary interest in infinite dimensional Banach manifolds where manifolds with corners are considered as well. They implicitly work with immersions and submersions as with maps locally equivalent to linear maps and similarly transversality is defined. In this setting, they prove advanced results such as a generalization of the multijet transversality theorem for manifolds with corners and deduce several approximation results. I used this book because of the proof of a version of the implicit function theorem contained therein.

(3) An article [10] where existence of transverse intersection $P \times_R Q$, where P, Q, R are manifolds with corners, is proven in the special case that Q is a regular submanifold of R. Transversality is defined just in the special case of a map and a regular submanifold. When $\partial R = \emptyset$ it reduces to our definition of transversality. The notion of a regular submanifold should correspond to the notion of a neatly embedded submanifold in this text. I adopted the formalism of local faces and the definition of a neat submersion from this article.

Except for the list above, I have not used any other known results about manifolds with corners. Results or concepts which I have not found in the literature and which I would like to highlight are: The concept of a neat immersion in Definition 15 and the corresponding part of the Local Form Theorem 1, the concept of a manifold with embedded faces in Definition 18, Theorem 9 which guarantees existence of a neat embedding of a compact manifold into the Euclidean corner, the concept of a vector field compatible with boundary in Definition 29, the concept of a neat system of compatible collar neighborhoods in Definition 35 and the corresponding existence Theorem 17.

2.1 Category of Manifolds with Corners

There are several definitions of a manifold with corners which are based on the same intuition and which generalize the standard definition of a manifold with and without boundary. A survey on the situation is given in [8]. We stick to a simple definition from [1] which is also mentioned in [4] and is at this point equivalent to the definition used in [8].

Definition 1 (Chart with Corners). *Let P be a topological space. A **chart with corners on** P is an ordered pair (U, φ) where $U \subset P$ is an open connected subspace and $\varphi : U \to V$ a homeomorphism onto an open subspace $V \subset \mathbb{R}^q_+$ for some $q \in \mathbb{N}_0$. Two charts with corners (U, φ) and (U', φ') on P are **compatible** if the transition map*

$$\begin{array}{rccc} \psi : & \varphi(U \cap U') & \to & \varphi'(U \cap U') \\ & x & \mapsto & \varphi'(\varphi^{-1}(x)) \end{array}$$

is a diffeomorphism.

The notion of a smooth map between subspaces of Euclidean spaces is defined in Definition 43 and the notion of a diffeomorphism of open subspaces of an Euclidean corner is defined in Definition 47 in the Appendix.

Definition 2 (Smooth Atlas with Corners). *A set $\mathcal{P}$ of charts with corners on P is called a **smooth atlas with corners on** P if the following conditions hold:*

(i) If $(U, \varphi), (U', \varphi') \in \mathcal{P}$ then they are compatible.

(ii) If $p \in P$ then there is $(U, \varphi) \in \mathcal{P}$ with $p \in U$.

*A smooth atlas with corners $\mathcal{P}$ is called **maximal** if the following holds:*

(iii) If a chart (U, φ) is compatible with every chart of $\mathcal{P}$ then $(U, \varphi) \in \mathcal{P}$.

Definition 3 (Manifold with Corners). *A* ***manifold with corners*** *is a Hausdorff, second-countable topological space P together with a maximal smooth atlas with corners on P. By charts on P we mean charts from the given maximal atlas.*

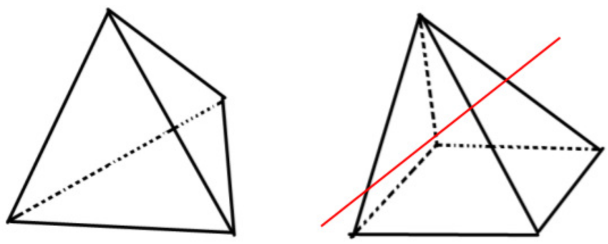

Figure 1: The 3-simplex on the left is a manifold with corners whereas the pyramid on the right not because of the top point.

Many proofs of basic facts for manifolds with corners are identical with the proofs for manifolds with or without boundary. This is the case of the following lemma from [4] whose proof depends only on basic topology and locality of smoothness of transition maps.

Lemma 1 (Existence of Maximal Atlas). *Let P be a topological space and $\mathcal{A}$ a smooth atlas on P. Then there is a unique maximal smooth atlas $\mathcal{P}$ on P such that $\mathcal{A} \subset \mathcal{P}$.*

Proof. Let $\mathcal{P}$ be the set of all charts compatible with each chart in $\mathcal{A}$. Let $(U_1, \varphi_1), (U_2, \varphi_2) \in \mathcal{P}$ be two charts. If $p \in U_1 \cap U_2$ then there is a chart $(V, \psi) \in \mathcal{A}$ containing p. It holds $\varphi_{12}(x) = ((\varphi_2 \circ \psi^{-1}) \circ (\psi \circ \varphi_1^{-1}))(x)$ for every $x \in \varphi_1(U_1 \cap U_2 \cap V)$; therefore, the restriction of the transition map on a neighborhood of x is smooth as a composition of smooth functions. It follows that $\varphi_{12} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is smooth since smoothness is a local property. Because $\varphi_{12}^{-1} = \varphi_{21}$, the transition map is a diffeomorphism and (U_1, φ_1) and (U_2, φ_2) are compatible. We get that all charts in $\mathcal{P}$ are compatible and cover P since $\mathcal{A} \subset \mathcal{P}$; hence, $\mathcal{P}$ is an atlas on P which is maximal by its definition. The maximal atlas $\mathcal{P}$ is a unique maximal atlas compatible with $\mathcal{A}$ as any other maximal atlas compatible with $\mathcal{A}$ is contained in $\mathcal{P}$ by definition. $\square$

EXAMPLE 1. (i) Let P be a manifold with corners and $U \subset P$ an open subspace. Charts on P restricted to U define a maximal smooth atlas with corners on U and make U into a manifold with corners. It is called the open submanifold with corners of P.

(ii) Let P and Q be manifolds with corners. The product $P \times Q$ has an atlas with corners consisting of product charts $(U \times V, \varphi \times \psi)$ where (U, φ) and (V, ψ) are charts on P and Q, respectively. The product chart is well defined since

$$\mathbb{R}^q_+ \times \mathbb{R}^p_+ = \mathbb{R}^{q+p}_+.$$

Lemma 1 asserts that an atlas of product charts can be completed to a maximal atlas which turns $P \times Q$ into a manifold with corners.

(iii) A countable disjoint union of manifolds with corners is a manifold with corners.

(iv) The space $\mathbb{R}^q_+$ and in turn every open subspace $U \subset \mathbb{R}^q_+$ with the maximal atlas generated by the identity map is a manifold with corners. All notions for the Euclidean corner from Appendix coincide with notions for $\mathbb{R}^q_+$ as for a manifold with corners.

(v) The standard simplex Δ^n is a manifold with corners. It is defined by

$$\Delta^n = \left\{ (t_0, \ldots, t_n) : t_i \in \mathbb{R}^{n+1}_+ : \sum_{i=0}^{n} t_i = 1 \right\}$$

(vi) The empty set $\emptyset$ and the one-point space $\{*\}$ are manifolds with corners: The empty space admits a unique chart $\emptyset_q : \emptyset \to \mathbb{R}^q_+$ which constitutes the unique maximal atlas $\{(\emptyset, \emptyset_q) : q \in \mathbb{N}_0\}$ for every $q \in \mathbb{N}_0$. The one-point space $\{*\}$ admits a unique chart $\varphi : \{*\} \to \mathbb{R}^0_+$.

Definition 4 (Smooth Map). *Let P, Q be manifolds with corners and $f : W \to Q$ a map from a subspace $W \subset P$. It is called* ***smooth on*** *W if for every $p \in W$ there is a chart (U, φ) on P containing p and a chart (V, ψ) on Q such that $f(U \cap W) \subset V$ and the following map is smooth*

$$\psi \circ f \circ \varphi^{-1} : \varphi(U \cap W) \to \psi(V) \tag{1}$$

Remark 1 (Basic Properties of Smooth Maps). (i) If $f : W \to Q$ is smooth then the map 1 is smooth with respect to any pair of charts (U, φ), (V, ψ) such that $f(U \cap W) \subset V$. This is easy to check using the fact that the transition functions are diffeomorphisms.

(ii) Smoothness is a local property: This means that f is smooth if and only if $f\big|_U : U \cap W \to V$ is smooth for all open submanifolds $U \subset P$ from some open cover of W where $V \subset Q$ is an open submanifold such that $f(U) \subset V$. The previous sentence is also true when "some" is replaced by "any". The proof is obvious from (i) and the definition of charts on U.

(iii) Composition of smooth maps is smooth. This follows from the situation on $\mathbb{R}^q_+$ since smoothness is a local property.

(iv) A smooth map is continuous. Continuity is a local property as well and it follows from the situation on $\mathbb{R}^q_+$ again.

Definition 5 (Diffeomorphism). *Let P and Q be manifolds with corners and $f : P \to Q$ a map. It is called a* ***diffeomorphism*** *if it is bijective and both $f : P \to Q$ and $f^{-1} : Q \to P$ are smooth.*

Remark 2 (A Chart is a Diffeomorphism). Let (U, φ) be a chart on a manifold with corners P. Then $\varphi : U \to \varphi(U)$ is a diffeomorphism of open submanifolds $U \subset P$ and $\varphi(U) \subset \mathbb{R}^q_+$.

Altogether, we have a category of manifolds with corners. Its objects are manifolds with corners, arrows are smooth maps, $\emptyset$ is the initial object and the one-point space $\{*\}$ the terminal object. The disjoint union is a countable coproduct and $P \times Q$ a finite product. Two manifolds with corners are isomorphic in this category if and only if they are diffeomorphic.

2.2 Strata and Faces

In this section we will study the basic local invariants of manifolds with corners. In addition to the standard notion of dimension, we define the depth of a point which characterizes type of the corner and reflects the fact that corners can not be smoothened by a smooth transformation.

LEMMA 2 (Local Invariants). *Let P be a manifold with corners and $p \in P$. If (U_1, φ_1), (U_2, φ_2) are two charts on P containing p then the coordinates $\varphi_1(p)$, $\varphi_2(p)$ have the same number of components, zero components, and non-zero components.*

Proof. The transition map $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ is a diffeomorphism of non-empty open subspaces of an Euclidean corner. The lemma follows from Lemma 40 in the Appendix. □

DEFINITION 6 (Dimension and Depth). *Let P be a manifold with corners and $p \in P$. We define the **dimension of** P **at** p to be the number of coordinates of p in one and any coordinate chart. We denote it by $\dim_P(p)$.*

*We also define the **depth of** p **in** P to be the number of zero coordinates of p in one and any coordinate chart. We denote it by $d_P(p)$.*

*If $\dim_P(p') = \dim_P(p)$ for every $p, p' \in P$ then we denote this number by $\dim(P)$ and call it the **dimension of** P.*

CONVENTION 1. We will assume all manifolds P in this text have well defined dimension $\dim(P)$. Because $\dim_P : P \to \mathbb{N}_0$ is locally constant; hence, constant on every connected component of P; this assumption means precisely that every two connected components of P have the same dimension.

DEFINITION 7 (Strata and Faces). *Let P be a manifold with corners and $k \in \mathbb{N}_0$. We set*

$$S^k(P) = \{p \in P : d_P(p) = k\}.$$

and let $\mathcal{S}^k(P)$ *to be the set of connected components of* $S^k(P)$*. We also set*

$$\mathcal{F}^k(P) = \{\overline{S} : S \in \mathcal{S}^k(P)\}.$$

Elements of $\mathcal{S}^k(P)$ *are called* k***-strata of*** P *and elements of* $\mathcal{F}^k(P)$ k***-faces of*** P*. The number* k *is called the* ***codimension of a stratum / face in*** P*. We set*

$$\mathcal{S}(P) = \bigcup_{l \in \mathbb{N}_0} \mathcal{S}^l(P) \qquad \textit{and} \qquad \mathcal{F}(P) = \bigcup_{l \in \mathbb{N}_0} \mathcal{F}^l(P)$$

and call their elements simply ***strata*** *and* ***faces****. For any* $p \in P$ *we also set*

$$\tilde{\mathcal{F}}_p(P) = \{F \in \mathcal{F}(P) : p \in F\}.$$

We define the ***boundary of*** P *and the* ***interior of*** P *to be the subspaces of* P *given by*

$$\partial P = \bigcup_{l \in \mathbb{N}} S^l(P) \qquad \textit{and} \qquad \mathrm{Int}(P) = S^0(P).$$

We define the ***abstract boundary of*** P *to be the topological disjoint union*

$$\tilde{\partial} P = \bigsqcup \mathcal{F}^1(P).$$

Lemma 3 (Diffeomorphism Preserves Strata and Faces). *Let* P *and* Q *be manifolds with corners and let* $f : P \to Q$ *be a diffeomorphism. Then for every* $p \in P$ *holds*

$$\dim_Q(f(p)) = \dim_P(p) \qquad \textit{and} \qquad d_Q(f(p)) = d_P(p).$$

In particular, f *preserves all strata and faces.*

Proof. If (U, φ) is a chart on P with $p \in U$ then $(f(U), \varphi \circ f^{-1})$ defines a chart on Q at $f(p)$ and the lemma follows from Lemma 2. □

EXAMPLE 2. (i) If $U \subset P$ is an open submanifold then $S^k(U) = U \cap S^k(P)$ for every $k \in \mathbb{N}_0$. Hence, strata / faces of U are precisely connected components of intersections of strata / faces of P with U.

(ii) Let P, Q be manifolds with corners and $P \times Q$ their product. Then

$$d_{P\times Q}((p,q)) = d_P(p) + d_Q(q)$$

holds from the construction for every $(p,q) \in P \times Q$. Consequently, we have

$$\mathcal{S}^k(P\times Q) = \{S \times S' : S \in \mathcal{S}^l(P), S' \in \mathcal{S}^m(Q), l+m=k; l,m \in \mathbb{N}_0\}$$

for all $k \in \mathbb{N}_0$. Because $\overline{S \times S'} = \overline{S} \times \overline{S'}$, the same conclusion holds for faces.

(iii) In case $P = \mathbb{R}^q_+$ the notions agree with Definition 45 in the Appendix. For an $x \in \mathbb{R}^q_+$ we see that

$$\tilde{\mathcal{F}}_x(\mathbb{R}^q_+) = \{\partial_I \mathbb{R}^q_+ : I \subset I_x\}.$$

It is immediate to check that $\mathcal{F}(\mathbb{R}^q_+)$ as well as $\mathcal{F}_x(\mathbb{R}^q_+)$ are closed on taking intersections since for any $I, J \subset \{1,\ldots,q\}$ holds

$$\partial_I \mathbb{R}^q_+ \cap \partial_J \mathbb{R}^q_+ = \partial_{I\cup J}\mathbb{R}^q_+.$$

In addition, if $\partial^0_I \mathbb{R}^q_+ \cap \partial_J \mathbb{R}^q_+ \neq \emptyset$ then $J \subset I$ and $\partial^0_I \mathbb{R}^q_+ \subset \partial_J \mathbb{R}^q_+$. Because every point lies in a stratum we get

$$\partial_J \mathbb{R}^q_+ = \bigcup_{J\subset I} \partial^0_I \mathbb{R}^q_+.$$

If $x \in \partial^0_I \mathbb{R}^q_+$ for some $I \subset \{1,\ldots,q\}$ then

$$\partial_I \mathbb{R}^q_+ = \bigcap \tilde{\mathcal{F}}_x(\mathbb{R}^q_+) \tag{2}$$

is the unique face with maximal codimension $d_{\mathbb{R}^q_+}(x)$ containing x.

(iv) If $P = \{*\}$ then the only stratum / face is the trivial 0-stratum / 0-face $\{*\}$. We suppose that $P = \emptyset$ has no stratum or face.

Next we are going to define a particular type of a chart which represents the local structure of a manifold with corners.

DEFINITION 8 (Regular Chart). *Let P be a manifold with corners and $p \in P$. A chart (U, φ) on P containing p is called a **regular chart at** p if*

(a) The image $\varphi(U)$ is a convex neighborhood of $\varphi(p)$ in $\mathbb{R}_+^{\dim(P)}$.

(b) Every $F \in \mathcal{F}(U)$ contains p. In other words, $\mathcal{F}(U) = \tilde{\mathcal{F}}_p(U)$.

(c) $\overline{U} \subset P$ is compact.

*In general, an open connected neighborhood $U \subset P$ of p which does not intersect any other face of P than those containing p will be called a **regular neighborhood** of p.*

REMARK 3. Without loss of generality we may assume that $\varphi(U)$ has shape of a coordinate ball centered at $\varphi(p)$ with a given radius.

LEMMA 4 (Countable Basis of Regular Charts). *Let P be a manifold with corners. Then there is a countable basis of regular charts.*

Proof. Second-countability implies there is a countable basis $\{B_i : i \in \mathbb{N}\}$. For every $i \in \mathbb{N}$ take a chart (U, φ) such that $B_i \subset U$ if such a chart exists. The set of charts $\mathcal{U}$ constructed in this way is a countable cover of P. For every $(U, \varphi) \in \mathcal{U}$ there is a countable basis of regular charts since pre-compact balls in $\mathbb{R}_+^q$ with rational radius and midpoint coordinates form a basis of $\varphi(U) \subset \mathbb{R}_+^q$. Their union over $\mathcal{U}$ gives the desired basis. $\square$

LEMMA 5 (Faces of a Regular Chart). *Let P be a manifold with corners, $p \in P$, and let (U, φ) be a regular chart at p. Then $\mathcal{F}(U) = \tilde{\mathcal{F}}_p(U)$ is closed on taking intersections of faces and $(\mathcal{F}(U), \cap)$ is isomorphic to $(\mathcal{F}_x(\mathbb{R}_+^q), \cap)$ where $x = \varphi(p)$, $q = \dim(P)$. The isomorphism is given by assigning to $F \in \mathcal{F}(U)$ the unique $\partial_I \mathbb{R}_+^q$ such that $\varphi(F) \subset \partial_I \mathbb{R}_+^q$.*

Proof. Because $\varphi : U \to \varphi(U)$ is a diffeomorphism it preserves faces by Lemma 3. We have that every non-empty face of $\varphi(U) \subset \mathbb{R}^q_+$ contains x. Since $\varphi(U)$ is convex, faces of $\varphi(U)$ are precisely intersections $\partial_I \mathbb{R}^q_+ \cap \varphi(U)$ for $I \subset I_x$. However, this is the definition of $\mathcal{F}_x(\mathbb{R}^q_+)$ from Remark 2. □

DEFINITION 9 (Local Faces). *Let P a manifold with corners and $p \in P$. Let (U, φ) be a regular chart at p and $F \in \mathcal{F}^k(U)$ for some $k \in \mathbb{N}_0$. Then we define the* ***local** k**-face of** P **at*** *p as a set-germ $[F]_p$ at p. We let $\mathcal{F}^k_p(P)$ to be the set of all local k-faces at p for $k \in \mathbb{N}_0$. We also set*

$$\mathcal{F}_p(P) = \bigcup_{k \in \mathbb{N}_0} \mathcal{F}^k_p(P).$$

REMARK 4 (Lattice of Set-Germs). (i) Recall that a set-germ at p is an equivalence class of subsets of P containing p, two subsets $S, S' \subset P$ being equivalent if there is an open neighborhood $U \subset P$ of p such that

$$S \cap U = S' \cap U.$$

We define a partial ordering $[S]_p \subset [S']_p$ if there is an open neighborhood $U \subset P$ of p such that $U \cap S \subset S'$. We also define the intersection and union

$$[S]_p \cap [S']_p = [S \cap S']_p \qquad \text{and} \qquad [S]_p \cup [S']_p = [S \cup S']_p.$$

If $f : P \to Q$ is a continuous map and $[S]_{f(p)}$ a germ at $f(p)$ we define the pull-back germ

$$f^*_p[S]_{f(p)} = [f^{-1}(S)]_p.$$

It is easy to check that the operations on set-germs above are well defined and give the set of set-germs structure of a bounded lattice with minimal element $[p]_p$ and maximal element $[P]_p$. The pull-back is then a morphism of bounded lattices.

(ii) Local faces $\mathcal{F}_p(P)$ generate a lattice if we consider their finite joins

$$[F_1]_p \cup \ldots \cup [F_n]_p.$$

By abuse of notation we will denote this lattice $\mathcal{F}_p(P)$ as well. Lemma 5 implies $\mathcal{F}_p(P) \simeq \mathcal{F}_{\varphi(p)}(\mathbb{R}^q_+)$ as lattices. Nevertheless, this is isomorphic to $\mathcal{F}_x(\mathbb{R}^q_+)$ for any $x \in \mathbb{R}^q_+$ with $d(x) = d_P(p)$ which we can easily see by renaming coordinates. Therefore, the lattice of local faces does not depend on a particular regular chart chosen and depends just on $d_P(p)$.

(iii) For every $[F]_p \in \mathcal{F}^k_p(P)$ there is precisely one $G \in \tilde{\mathcal{F}}^k_p(P)$ with $[F]_p \subset [G]_p$. On the other hand, if $G \in \tilde{\mathcal{F}}^k_p(P)$ then

$$[G]_p = \bigcup \{[F]_p : [F]_p \subset [G]_p, [F]_p \in \mathcal{F}^k_p(P)\}.$$

We will investigate manifolds where $\mathcal{F}_p(P) \simeq \tilde{\mathcal{F}}_p(P)$ in section 2.6.

Having the local faces, we may consider which properties of $\mathcal{F}_p(P) \simeq \tilde{\mathcal{F}}_x(\mathbb{R}^q_+)$ listed in (iii) of Remark 2 are valid for $\tilde{\mathcal{F}}_p(P)$ as well.

Lemma 6 (Intersection of a Stratum and a Face). *Let P be a manifold with corners. Let $S \in \mathcal{S}^k(P)$ and $S' \in \mathcal{S}^l(P)$ for some $k, l \in \mathbb{N}_0$ and denote $F = \overline{S}$. If $F \cap S' \neq \emptyset$ then $k \leq l$ and $S' \subset F$. If $k = l$ then $S = S'$.*

Proof. The set $F \cap S'$ is non-empty and closed in S'. In order to use connectedness of S' to show $F \cap S' = S'$ it suffices to prove that $F \cap S' \subset S$ is open. Let $p \in F \cap S'$. Then both $[F]_p$ and $[S']_p$ are local faces at p and it is enough to show that $[S']_p \subset [F]_p$. However, $d_P(p) = l$ and $[S']_p$ is the unique local face of highest codimension at p; thus, the inclusion holds due to the isomorphism $\mathcal{F}_p(P) \simeq \tilde{\mathcal{F}}_x(\mathbb{R}^q_+)$ and the Equation 2. Because the isomorphism preserves codimension as well we have $k \leq l$. Now let $k = l$. We have $S' \subset F$; hence, $S' \subset \overline{S} \cap d_P^{-1}(l) = S$ because S is closed in $d_P^{-1}(l)$. Therefore, $S = S'$ and the lemma is proven. $\square$

LEMMA 7 (Intersection of Two Faces). *Let P be a manifold with corners. Let $F \in \mathcal{F}^k(P)$ and $G \in \mathcal{F}^l(P)$ for some $k, l \in \mathbb{N}_0$. If there exists some $p \in F \cap G$ and a local l-face $[G']_p \subset [G]_p$ such that $[G']_p \subset [F]_p$ then $G \subset F$ and $k \leq l$. If $k = l$ then $G = F$.*

Proof. The fact that $[G]_p \subset [F]_p$ means by definition that there exists an open neighborhood $U \subset P$ of p such that $U \cap G \subset F$. If $S \in \mathcal{S}^l(P)$ is such that $G = \overline{S}$ then $S \cap F \neq \emptyset$ and the previous lemma applies. □

REMARK 5 (Structure of Faces). If $F \in \mathcal{F}^k(P)$ then Lemma 6 asserts F is a union of a unique k-stratum S and some other strata of strictly greater codimension. In $\mathbb{R}^q_+$, every point in depth greater than k is contained in a $(k+1)$-face. Therefore, for every $p \in F \backslash S$ there is a local $(k+1)$-face at p which is contained in a global $(k+1)$-face. It follow that F can be expressed also as a union of S and certain $(k+1)$-faces of P.

In [8] and [10] they use a slightly different definition of a manifold with corners where a chart at p is a homeomorphism ψ of a neighborhood of p and a neighborhood of $\psi(p) = 0$ in

$$\mathbb{R}^q_k = \mathbb{R}^k \times \mathbb{R}^{q-k}_+$$

for some $0 \leq k \leq q$ and $q \in \mathbb{N}_0$. However, if we set $k = d_P(p)$ we see that a regular chart (U, φ) at p is diffeomorphic to such a neighborhood of 0 in $\mathbb{R}^q_k$ by a simple translation in $\mathbb{R}^q$. On the other hand, a ball around 0 in $\mathbb{R}^q_k$ can be translated to an open subset of $\mathbb{R}^q_+$. Using this it is immediate to show that the two definitions of manifolds with corners are equivalent in the sense of equivalence of categories. Since $\mathbb{R}^q = \mathbb{R}^q_q$ and $\mathbb{H}^q = \mathbb{R}^q_{q-1}$ it follows that the categories of manifolds without boundary / manifolds with boundary are equivalent to subcategories of the category of manifolds with corners with $\partial P = \emptyset$ / $d_P(p) \leq 1$ for all $p \in P$.

The spaces $\mathbb{R}^q_k$ are pairwise non-diffeomorphic as manifolds with corners. Nevertheless, for every $k < q$ there is a homeomorphism of $\mathbb{R}^q_k$ and $\mathbb{R}^q_{k+1}$ given, for example, by identifying k-th and $(k+1)$-th coordinate of $\mathbb{R}^q_k$ with the real and imaginary axis of $\mathbb{C}$ and applying the complex square $z \mapsto z^2$. It follows that $\mathbb{R}^q_+ = \mathbb{R}^q_0 \simeq \ldots \simeq \mathbb{R}^q_{q-1} = \mathbb{H}^q$ as topological spaces. Therefore, a manifold with corners is a topological manifold with boundary.

2.3 Tangent Space and Differential

In this section we will define the tangent space and the differential. To define the tangent space at points in the interior and in the boundary consistently we choose a definition from [6]. It is also possible to define tangent vectors as derivatives on smooth functions as in [8] or [4]. However, the identification with the equivalence classes of smooth curves having the same derivative does not give us what we want at the boundary points.

Definition 10 (Tangent Vectors). *Let P be a manifold with corners and $p \in P$. Consider the set of ordered pairs $(v, (U, \varphi))$ where $v \in \mathbb{R}^{\dim(P)}$ is a vector and (U, φ) a chart on P which contains p. Two such pairs $(v, (U, \varphi))$ and $(v', (U', \varphi'))$ are equivalent if*

$$\mathrm{d}(\varphi' \circ \varphi^{-1})(\varphi(p))v = v'.$$

We denote the set of equivalence classes T_pP and equip it with the obvious vector space structure which makes T_pP isomorphic to $\mathbb{R}^{\dim(P)}$. We call T_pP the ***tangent space to P at p****. Elements of T_pP are called* ***tangent vectors at*** *p. We define a set*

$$TP = \bigsqcup_{p \in P} T_pP$$

and call it the ***tangent space to*** *P.*

Remark 6. The relation on pairs is transitive because of the chain rule stated in Remark 36 in the Appendix.

Convention 2. When a chart (U, φ) around p is chosen we fix the identification $T_pP \simeq \mathbb{R}^q$ and refer to the tangent vector with representant $(v, (U, \varphi))$ just as to v and to the tangent space just at to $\mathbb{R}^q$.

Definition 11 (Differential). *Let $f : P \to Q$ be a smooth map and $p \in P$. Let (U, φ) and (V, ψ) be charts on P and Q with $f(U) \subset V$. The linear map* *defined as*

$$\begin{array}{rccc} \mathrm{d}f_p : & T_pP & \to & T_{f(p)}Q \\ & v & \mapsto & \mathrm{d}(\psi \circ f \circ \varphi^{-1})_{\varphi(p)} v \end{array}$$

is called the ***differential of*** f ***at*** p.

REMARK 7. Differential of a smooth map on an open subspace of $\mathbb{R}^q_+$ is well defined as in Definition 44 in the Appendix. The definition above does not depend on the pair of charts because of the chain rule.

DEFINITION 12 (Space of Tangent Vectors to Curves). *Let P be a manifold with corners and $p \in P$. We define $C_pP \subset T_pP$ to be the set of tangent vectors $\gamma'(0)$ to smooth curves $\gamma : (-\varepsilon, \varepsilon) \to P$. Similarly, we define $T_p^+P \subset T_pP$ to be the set of tangent vector to smooth curves $\gamma : [0, \varepsilon) \to P$ with $\gamma(0) = p$.*

LEMMA 8 (Preservation of Tangent Vectors to Curves). *Let P be manifold with corners and $p \in P$. Then $C_pP \subset T_pP$ is a vector subspace with*

$$\operatorname{codim}_{T_pP}(C_pP) = d_P(p) \qquad \textit{and} \qquad C_pP = T_p^+P \cap (-T_p^+P).$$

If Q is an other manifold with corners and $f : P \to Q$ a smooth map then

$$\mathrm{d}f_p(C_pP) \subset C_{f(p)}Q \qquad \textit{and} \qquad \mathrm{d}f(T_p^+P) \subset T_{f(p)}^+Q.$$

Proof. The proof follows from the local Lemma 38 in the Appendix. □

EXAMPLE 3. (i) Is $U \subset P$ is an open submanifold then $T_pU = T_pP$, $T_p^+U = T_p^+P$, and $C_pU = C_pP$ for every $p \in U$.

(ii) If P, Q are manifolds with corners then $T_{(p,q)}(P \times Q) = T_pP \oplus T_qQ$, $T_{(p,q)}^+(P \times Q) = T_p^+P \oplus T_q^+Q$, and $C_{(p,q)}(P \times Q) = C_pP \oplus C_qQ$. for every $(p, q) \in P \times Q$.

(iii) Tangent space to $\{*\}$ at the only point $*$ is the zero vector space. We do not need to define tangent space to an empty set as $\emptyset$ has no point.

Definition 13 (Smooth Vector Field). *Let P be a manifold with corners. A map $X : P \to TP$ is called a **smooth vector field on** P if for every $p \in P$ holds $X(p) \in T_pP$ and there is a chart (U, φ) such that $p \in U \mapsto X(p) \in \mathbb{R}^{\dim(P)} \overset{\varphi}{\simeq} T_pP$ is smooth.*

2.4 Differential Properties of Smooth Maps

In this section we are going to elaborate on the notions of immersion, submersion and transversality on manifolds with corners. Our notion of a neat submersion and transversality is similar to the one in [10].

Lemma 9 (Depth-Rank Inequality). *Let P,Q be manifolds with corners and $f : P \to Q$ a smooth map. Then for every $p \in P$ holds*

$$\dim_Q(f(p)) - d_Q(f(p)) \geq \mathrm{rk}(\mathrm{d}f_p) - d_P(p)$$

Proof. See Lemma 39 in the Appendix. □

Definition 14 (Immersion and Submersion). *Let P, Q be manifolds with corners and $f : P \to Q$ a smooth map. Let $p \in P$ be such that $\mathrm{d}f_p : T_pP \to T_{f(p)}Q$ has maximal rank. Then f is called an **immersion at** p if $\dim(P) \leq \dim(Q)$ and a **submersion at** p if $\dim(P) \geq \dim(Q)$.*

Remark 8 (Properties of Immersions and Submersions). (i) Let $f : P \to Q$ and $g : Q \to R$ be arbitrary maps of manifolds with corners. If g is an immersion / f is a submersion then f is smooth / g is smooth if and only if $g \circ f$ is smooth: One direction is clear. It is a local statement so that we may prove it just for open subspaces of $\mathbb{R}^q_+$ and assume the smooth maps are restrictions of their smooth extensions. By the implicit function theorem for $\mathbb{R}^q$ the smooth extension of g / f can be composed with a diffeomorphism from the left / right so that it becomes a linear injection ι / projection π. If h is a smooth extension of $g \circ f$ we get a smooth extension of f / g as $\pi \circ h$ / $h \circ \iota$. Therefore, f / g is smooth.

(ii) Let $f : P \to Q$ be a smooth bijection such that $\mathrm{d}f_p : T_pP \to T_{f(p)}Q$ is an isomorphism for all $p \in P$. Then $f \circ f^{-1} = \mathrm{id}_Q$ and $f^{-1} \circ f = \mathrm{id}_P$ imply that $f^{-1} : Q \to P$ is smooth since f is both an immersion and a submersion. Therefore, f is a diffeomorphism. Note that the single fact that $\mathrm{d}f_p$ is an isomorphism does not imply f restricts to a diffeomorphism of an open neighborhood U of p. The problem is that $f(U)$ does not have to be an open subspace. For instance, the translation $x \mapsto x + a$ of $\mathbb{R}^q_+$ by a vector $0 \neq a \in \mathbb{R}^q_+$ is not a diffeomorphism of open subspaces of $\mathbb{R}^q_+$ near any point. It can be avoided by requiring that f preserves the boundary. See the Inverse Function Theorem 20 in the Appendix.

(iii) Composition of immersions / submersions is an immersion / submersion by the chain rule.

Definition 15 (Neat Immersions and Submersions). *Let P, Q be manifolds with corners, $p \in P$ and let $f : P \to Q$ be a smooth map. Set $q = f(p)$ and let $\mathcal{F}^1_p(P) = \{[F_1]_p, \ldots, [F_{d_P(p)}]_p\}$, $\mathcal{F}^1_q(Q) = \{[F'_1]_q, \ldots, [F'_{d_Q(q)}]_q\}$. Then f is called a* ***neat immersion at*** *p if*

(i) f is an immersion at p.

(ii) The map $T_pP/C_pP \to T_{f(p)}Q/C_{f(p)}Q$ induced by $\mathrm{d}f_p$ is injective.

(iii) It holds $[F_i]_p \subset f^*_p[F'_i]_{f(p)}$ for $i = 1, \ldots, d_P(p)$ and $[P]_p \subset f^*_p[F'_i]_{f(p)}$ for $i = d_P(p) + 1, \ldots, d_Q(f(p))$.

It is called a ***neat submersion at*** *p if*

(i) f is a submersion at p.

(ii) The map $C_pP \to C_{f(p)}Q$ induced by $\mathrm{d}f_p$ is surjective.

(iii) It holds $[F_i]_p \subset f^*_p[F'_i]_{f(p)}$ for every $i = 1, \ldots, d_Q(f(p))$.

Theorem 1 (Local Form of a Neat Immersions and Submersions). *Let P, Q be manifolds with corners, $p \in P$ and let $f : P \to Q$ be a smooth map. Then the following equivalences hold:*

(a) f is a neat immersion at p if and only if for every chart (U, φ) on P at p there is a chart (W, ψ) on Q at $f(p)$ such that $\psi \circ f \circ \varphi^{-1}$ equals the inclusion

$$x \in \mathbb{R}_+^{\dim(P)} \mapsto (x, a) \in \mathbb{R}_+^{\dim(Q)} = \mathbb{R}_+^{\dim(P)} \times \mathbb{R}_+^{\dim(Q)-\dim(P)}$$

for some $a \in \mathbb{R}_+^{\dim(Q)-\dim(P)}$ on a neighborhood of $\varphi(p)$

(b) f is a neat submersion at p if and only if for every chart (W, ψ) on Q at $f(p)$ there is a chart (U, φ) on P at p such that $\psi \circ f \circ \varphi^{-1}$ equals the projection

$$(x, y) \in \mathbb{R}_+^{\dim(P)} = \mathbb{R}_+^{\dim(Q)} \times \mathbb{R}_+^{\dim(P)-\dim(Q)} \mapsto x \in \mathbb{R}_+^{\dim(Q)}$$

on a neighborhood of $\varphi(p)$.

Proof. See theorems 21 and 22 in the Appendix. □

REMARK 9 (Properties of Neat Immersions and Submersions). (i) Composition of neat immersions/submersions is a neat immersion/submersion.

(ii) If $f : P \to Q$ is a (neat) immersion/submersion at $p \in P$ then there is an open neighborhood $U \subset P$ of p such that f is a (neat) immersion/submersion on U: In case of immersion / submersion it follows from the fact that having a maximal rank is a topologically stable property. If the maps are neat then it follows from the previous theorem.

THEOREM 2 (Inverse Function Theorem). *Let P, Q be manifolds with corners such that* $\dim(P) = \dim(Q)$ *and let $f : P \to Q$ be a smooth map. Let $p \in P$. If f is immersive at p and $[\partial P] \subset f_p^*[\partial Q]$ then there is an open neighborhood $U \subset P$ of p such that $f(U) \subset Q$ is open and $f : U \to f(U)$ is a diffeomorphism of open submanifolds. In particular, if f is an injective immersion on P and $f(\partial P) \subset \partial Q$ then $f(P) \subset Q$ is open and $f : P \to f(P)$ is a diffeomorphism.*

Proof. Follows from Theorem 20 in the Appendix. □

The following definition from [1] generalizes the notion of transversality to maps from manifolds with corners to manifolds without boundary.

DEFINITION 16 (Transversality). *Let P, Q, M be manifolds with corners, $\partial M = \emptyset$, $p \in P$ and let $f : P \to M$, $g : Q \to M$ be smooth maps. We say that f* ***is transverse to*** *g* ***at*** *$p \in P$ and write $f \pitchfork_p g$ if for every $q \in Q$ such that $f(p) = g(q) = m \in M$ holds*

$$\mathrm{d}f_p\,(C_pP) + \mathrm{d}g_q\,(C_qQ) = T_mM$$

We also say that f ***and*** *g* ***are transverse along*** *$S \subset P$ and write $f \pitchfork_S g$ if $f \pitchfork_p g$ for every $p \in S$. If $S = P$ then we say f* ***is transverse to*** *g and write simply $f \pitchfork g$. We also denote*

$$P \times_M Q = \{(p, q) \in P \times Q : f(p) = g(q) = m\}.$$

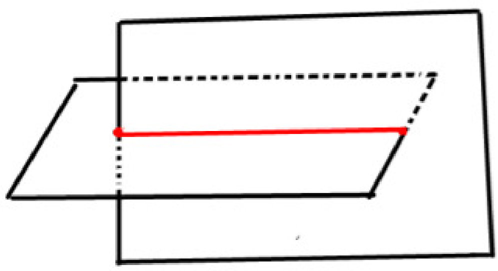

Figure 2: Two transverse quadrilaterals in $\mathbb{R}^3$. Note that two edges can not intersect transversally.

We might wish to replace boundaryless M by a general manifold with corners. In that case, however, all intersection points $m = f_1(p_1) = f_2(p_2)$ would lie in $\mathrm{Int}(M)$ as a consequence of Lemma 9. To get a non-trivial generalization which allows intersections within the boundary ∂M a more involved transversality condition would have to be introduced. See [10] for an approach closest to our approach or [8] for a different approach with different definitions.

2.5 Embedded Submanifolds

In this section we will see that the notions of a neat immersion and a neat submersion may be used to generalize theorems about submanifolds valid for manifolds without boundary to manifolds with corners; mostly, by simple addition of the word "neat". We will end up this section with the transverse intersection theorem which is just the case.

DEFINITION 17 (Embedded Submanifold). *Let P be a manifold with corners and $S \subset P$ a subspace. Then S is called a* ***(neatly) embedded submanifold*** *of P if it has a manifold with corners structure such that the inclusion $\iota : S \hookrightarrow P$ is a* ***(neat) embedding****; this means, it is a homeomorphism onto its image and a (neat) immersion.*

LEMMA 10 (Uniqueness of the Submanifold Structure). *Let P be a manifold with corners and $S \subset P$ an embedded submanifold. Then the manifold with corners structure on S such that $\iota : S \hookrightarrow P$ is an embedding is unique.*

Proof. Assume there are two maximal atlases $\mathcal{S}$, $\mathcal{S}'$ on S such that the inclusion is an embedding. Let $(W, \psi) \in \mathcal{S}$, $(W', \psi') \in \mathcal{S}'$, $p \in W \cap W'$ and let (U, φ) be a chart on P containing p. It suffices to prove that the transition function $\psi' \circ \psi^{-1}$ is smooth on $\psi(U \cap W \cap W')$. The coordinate representation $\tilde{\iota} = \varphi \circ \iota \circ \psi'^{-1}$ is an immersion and $\tilde{\iota} \circ (\psi' \circ \psi^{-1}) = \varphi \circ \iota \circ \psi^{-1}$ is smooth; therefore, the transition function is smooth by (i) of remark 8. □

LEMMA 11 (Image of an Embedding is an Embedded Submanifold). *Let P, Q be manifolds with corners and $f : P \to Q$ a (neat) embedding. Then $f(P) \subset Q$ is a (neatly) embedded submanifold and $f : P \to f(P)$ is a diffeomorphism.*

Proof. We get a maximal atlas on $f(P)$ by taking charts $(f(U), \varphi \circ f^{-1})$ where (U, φ) is a chart on P. This makes $f(P)$ diffeomorphic to P through

f. The inclusion $\iota : f(P) \hookrightarrow Q$ factors as $f(P) \xhookrightarrow{f^{-1}} P \xrightarrow{f} Q$ where f^{-1} is a diffeomorphism; hence, ι is a (neat) embedding if and only if f is. $\square$

Remark 10 (Useful Properties Of Embeddings). (i) Let P, Q be manifolds with corners such that $Q \subset P$ and $\dim(P) = \dim(Q)$. If $Q \subset P$ is an embedded submanifold then Lemma 9 implies $\mathrm{Int}(Q) \subset \mathrm{Int}(P)$. It follows from the standard Inverse Function Theorem that $\mathrm{Int}(Q) \hookrightarrow \mathrm{Int}(P)$ is an open submanifold. If Q is neatly embedded submanifold of P then $Q \subset P$ is an open submanifold by Theorem 1.

(ii) Let P, Q be manifolds with corners, $A \subset P$, $B \subset Q$ subspaces and $f : A \to B$ a smooth map onto B with a smooth inverse $g : B \to A$. If A is an embedded submanifold of P then f is a smooth embedding. This can be seen in coordinates by differentiating composition $\tilde{g} \circ \tilde{f}$ of smooth extensions using the chain rule. Note that it can not be argued directly since we can not differentiate g on an arbitrary subspace; it is not well defined. By the previous lemma we get that $B = f(A)$ is an embedded submanifold of Q and f is a diffeomorphism.

Theorem 3 (Slice Criterion for Submanifolds). *Let P be a manifold with corners and $S \subset P$ a subspace. Then $S \subset P$ is a (neatly) embedded submanifold if and only if for every $p \in S$ there is a chart (U, φ) on P at p, an open subspace $V \subset \mathbb{R}^q_+$ for some $q \in \mathbb{N}_0$, and a (neat) embedding $L : V \to \varphi(U)$ such that*

$$\varphi(U \cap S) = L(V).$$

In the neat case, L can be replaced by an inclusion $L : x \in \mathbb{R}^q_+ \mapsto (x, a) \in \mathbb{R}^{\dim(P)}_+$, $a \in \mathbb{R}^{\dim(P)-q}_+$ in the previous sentence and the equivalence still holds.

Proof. The "only if" part is clear: For a $p \in S$ we take charts (U, φ) on P and (W, ψ) on S both containing p and such that $W = U \cap S$. We set

$q = \dim(S)$, $V = \psi(W)$ and $L = \varphi \circ \iota \circ \psi^{-1}$ where $\iota : S \hookrightarrow P$ is the inclusion.

As for the "if" part, the aim is to construct an atlas on S such that ι looks locally like L. Given (U, φ), V and L at $p \in S$ we set $W = U \cap S$ and let $\psi : W \to V$ be the unique map such that $L \circ \psi = \varphi \circ \iota$ on W. It is a homeomorphism since $L : V \to \varphi(W)$ is. Therefore, the pair (W, ψ) is a chart on S. Let (W', ψ') be another chart constructed from L' in this way. The transition function $\psi' \circ \psi^{-1}$ is a homeomorphism of open subspaces of $\mathbb{R}^q_+$. Because $L' \circ (\psi' \circ \psi^{-1}) = \varphi' \circ \iota \circ \psi^{-1} = (\varphi' \circ \varphi^{-1}) \circ (\varphi \circ \iota \circ \psi^{-1}) = (\varphi' \circ \varphi^{-1}) \circ L$ is smooth and L' is an immersion we get that $\varphi' \circ \varphi^{-1}$ is smooth and similarly for the inverse. The charts are therefore compatible and generate a maximal atlas by Lemma 1. The inclusion ι looks locally like L; thus, it is a (neat) embedding. The neat version of the theorem with linear injections follows from Theorem 1. □

Example 4 (Strata are Submanifolds). Let $S \in \mathcal{S}^k(P)$, $p \in P$ and let (U, φ) be a regular chart at p. Then $\varphi(U \cap S) \subset \partial^0_{I_x} \mathbb{R}^q$ is an open subspace where $x = \varphi(p)$, $q = \dim(P)$. This open subspace can be identified with an open subspace of $\mathbb{R}^{q-k}$ by the projection $z \mapsto z^{I^c_x}$. It follows from the previous Theorem 3 that $S \subset P$ is a neatly embedded submanifold without boundary. Clearly $\operatorname{codim}_P(S) = k$.

Lemma 12 (Regular Level Sets). *Let P be a manifold with corners and $S \subset P$ a subspace. Then S is a neat submanifold of P if and only if for every $p \in S$ there is an open neighborhood $W \subset P$ of p, a manifold with corners Q and a smooth map $f : W \to Q$ which is a neat submersion on $S \cap W$ and*

$$f^{-1}(q) = S \cap W \tag{3}$$

for some $q \in Q$. In addition, for every $p \in S \cap W$ it holds

$$\operatorname{codim}_P(S) = \dim(Q) \qquad \textit{and} \qquad d_S(p) = d_P(p) - d_Q(q)$$

Proof. As for the "only if" part, let (U, φ) be a chart on P at p such that

$$\varphi(U \cap S) = \{(x, y) \in \varphi(U) : x \in \mathbb{R}_+^{\dim(P)-s}, y = a\}$$

for an $s \in \mathbb{N}_0$ and $a \in \mathbb{R}_+^s$ from Theorem 3. We set $W = U$, $Q = \mathbb{R}_+^s$ and define $f = \pi \circ \varphi : W \to Q$ where $\pi : \mathbb{R}_+^{\dim(P)} \to \mathbb{R}_+^s$ is the standard projection. It is easy to see that f a neat submersion on U as a composition of neat submersions and that Equation 3 holds. The sum-of-depths formula holds as well since $d((x, a)) = d(x) + d(a)$.

Let us prove the "if" part: Let $f : W \to Q$ be a neat submersion at $p \in W \cap S$. Let (U', φ') be a chart on Q at q and (U, φ) a chart on W at p from Theorem 1 such that $\varphi' \circ f \circ \varphi^{-1} = \pi$ on $\varphi(U)$ where $\pi : \mathbb{R}_+^{\dim(P)} \to \mathbb{R}_+^{\dim(Q)}$ is the canonical projection. If we denote $a = \varphi'^{-1}(q)$ we have

$$\varphi(U \cap S) = \varphi(f^{-1}(q) \cap U) = \{z \in \varphi(U) : \pi(z) = a\} = \varphi(U) \cap L(V)$$

where $V = \{x \in \mathbb{R}_+^{\dim(P)-\dim(Q)} : (x, a) \in \varphi(U)\}$ is open and $L : V \to \varphi(U)$ sends $x \mapsto (x, a)$. Theorem 3 finishes the proof. $\square$

We are going to prove the transverse intersection theorem. To achieve this, we will use the following lemma.

Lemma 13 (Equivalent Transversality Conditions). *Let P, Q, M be manifolds with corners, $\partial M = \emptyset$, and let $f : P \to M$, $g : Q \to M$ be smooth maps. Then the following conditions are equivalent:*

(a) f is transverse to g.

(b) $S^k f \pitchfork S^l g$ for every $k, l \in \mathbb{N}_0$ where $S^k f = f\big|_{S^k(P)}$ and $S^l g = g\big|_{S^k(Q)}$.

(c) $(f \times g) \pitchfork \Delta$ *where* $\Delta : M \hookrightarrow M \times M$ *is the diagonal embedding.*

Proof. Conditions (a) and (b) are equivalent since

$$\mathrm{d}f_p(C_pP) = \mathrm{d}\left(S^k f\right)_p (T_p S^k P) \quad \text{and} \quad \mathrm{d}g_q(C_qQ) = \mathrm{d}\left(S^k g\right)_q (T_p S^l Q)$$

whenever $p \in S^k P$, $q \in S^l Q$.

In order to prove that (a) and (c) are equivalent consider that

$$\mathrm{d}\Delta_m(T_mM) = \{(v, v) : v \in T_mM\}$$

for every $m \in M$. Let $f(p) = g(q) = m$ for some $p \in P$, $q \in Q$. If (a) holds and $(a, b) \in T_mM \oplus T_mM$ then there are $a_1, b_1 \in \mathrm{d}f_p(C_pP)$, $a_2, b_2 \in \mathrm{d}g_q(C_qQ)$ such that $a = a_1 + a_2$, $b = b_1 + b_2$. Therefore, (a, b) decomposes as

$$(a, b) = (a_1, a_1) + (b_2, b_2) + (a_2 - b_2, b_1 - a_1)$$

where $b_1 - a_1 \in \mathrm{d}f_p(C_pP)$, $a_2 - b_2 \in \mathrm{d}g_q(C_qQ)$ and (c) holds. On the other hand, if (c) holds then for an $a \in T_mM$ we get a decomposition

$$(a, 0) = (v, v) + (c, d)$$

where $v \in T_mM$, $c \in \mathrm{d}f_p(C_pP)$, $d \in \mathrm{d}g_q(C_qQ)$; hence, $a = c - d \in \mathrm{d}f_p(C_pP) + \mathrm{d}g_q(C_qQ)$ and (a) holds. The lemma is hereby proven. $\square$

Theorem 4 (Transverse Intersection Theorem). *Let P, Q, M be manifolds with corners, $\partial M = \emptyset$, and let $f : P \to M$, $g : Q \to M$ be smooth maps. If $f \pitchfork g$ then $P \times_M Q \subset P \times Q$ is a neatly embedded submanifold of $P \times Q$ which is a closed subset. In addition, it holds*

$$\dim(P \times_M Q) = \dim(P) + \dim(Q) - \dim(M)$$

and

$$d_{P \times_M Q}((p, q)) = d_P(p) + d_Q(q)$$

for every $(p, q) \in P \times_M Q$.

Proof. Let $(p, q) \in P \times_M Q$ and $f(p) = f(q) = m \in M$. Let (W, ψ) be a chart on M at m and define a smooth map $v : W \times W \to \mathbb{R}^{\dim(M)}$ as $v = r \circ (\psi \times \psi)$ where $r(x, y) = x - y$ for $x, y \in \mathbb{R}^{\dim(M)}$. Then $\Delta(W) = v^{-1}(0)$ and $\mathrm{Im}(\mathrm{d}\Delta_m) = \mathrm{Ker}(\mathrm{d}v_{(m,m)})$. By Lemma 13 it holds $\mathrm{d}f_p\,(C_pP) + \mathrm{d}g_q\,(C_qQ) = T_mM$ if and only if

$$\mathrm{d}f_p(C_pP) \oplus \mathrm{d}g_q(C_qQ) + \mathrm{Im}(\mathrm{d}\Delta_m) = T_mM \oplus T_mM.$$

However, $\mathrm{Im}(\mathrm{d}\Delta_m) = \mathrm{Ker}(\mathrm{d}v_{(m,m)})$ and we see the transversality condition is satisfied if and only if $\mathrm{d}v_{(m,m)}(\mathrm{d}f_p(C_pP) \oplus \mathrm{d}g_q(C_qQ)) = T_{v(m)}\mathbb{R}^{\dim(M)}$. In other words, if and only if the map

$$h = v \circ (f \times g) \tag{4}$$

defined on an open neighborhood $U \subset P \times Q$ of (p, q) is a neat submersion at (p, q). Note that no other conditions of neatness have to be checked since the target manifold is a manifold without boundary. By (ii) of Remark 9 we may assume it is a neat submersion on U after possible shrinking of U. Because $\Delta(W) = v^{-1}(0)$ we have that

$$P \times_M Q \cap U = U \cap h^{-1}(0)$$

and lemma 12 applies. The depth formula holds too since $\mathrm{R}^{\dim(M)}$ is boundaryless. $\square$

Note that the proof is the same as the proof for boundaryless manifolds. The only difference is that we work with neat submersions instead of plain submersions. However, if we wanted to generalize to the case $\partial M \neq \emptyset$ using some reasonable definition of transversality which does not imply that the intersection lies necessarily in $\mathrm{Int}(M)$, a different proof would have to be thought up since the diagonal embedding is not necessary neat. Correspondingly, this theorem is proven for $\partial M \neq \emptyset$ in the case that one of the maps is a neat immersion in [10] .

LEMMA 14 (Transversality Holds on Open Subset). *Let P, Q, M be manifolds with corners such that $\partial M = \emptyset$ and Q is compact. Let $p \in P$ and let $f : P \to M$, $g : Q \to M$ be smooth maps. If $f \pitchfork_{\{p\}} g$ then there is an open neighborhood $U \subset P$ of p such that $f \pitchfork_U g$.*

Proof. For every $q \in Q$ we have either $f(p) = g(q)$ or $f(p) \neq g(q)$. In the first case, the transversality condition is satisfied at (p, q); hence, it is satisfied on an open neighborhood $U_p \times V_q \subset P \times Q$ because it is locally equivalent to map 4 being a neat submersion at (p, q) in the proof of the previous theorem. If $f(p) \neq g(q)$ then there is an open neighborhood $U_p \times V_q \subset P \times Q$ such that $f(p') \neq g(q')$ for all $(p', q') \in U_p \times V_q$. Because Q is compact there is a finite set of points (p_i, q_i), $i = 1, \ldots, n$ such that $Q \subset \bigcup_{i=1}^n V_{q_i}$. If we set $U = \bigcap_{i=1}^n U_{p_i}$ then any $(p', q') \in U \times Q$ lies in one of $U_{p_i} \times V_{q_i}$ for which either $U_{p_i} \times V_{q_i} \cap P \times_M Q = \emptyset$ or the transversality condition is satisfied. Therefore, U is the desired neighborhood. $\square$

REMARK 11 (Strata of Transverse Intersection). Because the embedding $P \times_M Q \subset P \times Q$ is depth-preserving we may write

$$S^k(P \times_M Q) = \bigsqcup_{\substack{l,m \in \mathbb{N}_0 \\ l+m=k}} S^l(P) \times_M S^m(Q) = \bigsqcup_{\substack{S \in \mathcal{S}^l(P), S' \in \mathcal{S}^m(Q) \\ l+m=k}} S \times_M S'$$

Different $S \times_M S'$ are separated in $P \times_M Q$ since different $S \times S'$ are separated in $P \times Q$ as different strata of the same codimension. This holds because of Lemma 6. Therefore, the equality above holds on the topological level too. We see that k-strata of $P \times_M Q$ are precisely connected components of $S \times_M S'$, $S \in \mathcal{S}^l(P), S' \in \mathcal{S}^m(Q)$, $l + m = k$.

2.6 Manifolds with Embedded Faces

In this section we are going to define a manifold with embedded faces P which is a more reasonable object to work with than a manifold with corners. We will see faces of P are themselves manifolds with embedded faces and neatly embed into P.

DEFINITION 18 (Manifold with Embedded Faces). *Let P be a manifold with corners such that every $F \in \mathcal{F}^1(P)$ is an embedded submanifold with corners of P. Then P is called a* ***manifold with embedded faces****.*

THEOREM 5 (Equivalent Conditions for a Manifold with Embedded Faces). *Let P be a manifold with corners. Then the following conditions are equivalent:*

(a) P is a manifold with embedded faces.

(b) Every $F \in \mathcal{F}^1(P)$ is a neatly embedded submanifold with corners of P.

(c) For every $p \in P$ it holds $\mathcal{F}^1_p(P) = \{[G]_p : G \in \mathcal{F}^1(P)\}$. In other words, local 1-faces correspond to global 1-faces at p.

Proof. We will prove the implications (b) to (a), (a) to (c), and (c) to (b). The first implication is clear from the definition.

As for the second implication (a) to (c), (iii) of remark 4 asserts that for every $[F]_p \in \mathcal{F}^1_p(P)$ there is a unique $G \in \mathcal{F}^1(P)$ such that $[F]_p \subset [G]_p$ and that $[G]_p$ is a union of certain local 1-faces at p. Therefore, condition (c) is satisfied at p if and only if no $G \in \mathcal{F}^1(P)$ contains more than one local 1-face at p. In order to get a contradiction, suppose there is a $G \in \mathcal{F}^1(P)$ and distinct $[F]_p, [F']_p \in \mathcal{F}^1_p(P)$ such that $[F]_p, [F']_p \subset [G]_p$. Condition (a) asserts G is an embedded submanifold with corners. Let $S \in \mathcal{S}^1(P)$ be such that $G = \overline{S}$. From what we know, $S \subset P$ is an embedded submanifold without boundary. At the same time it is an open submanifold of G. However, these smooth structures are the same by Lemma 10. Therefore, $\dim(G) = \dim(S) = \dim(P) - 1$ and $S \subset \mathrm{Int}(G)$. Lemma 9 applied to the inclusion $G \hookrightarrow P$ gives $1 \geq d_P(p') - d_G(p')$ for every $p' \in G$; hence, $S^2(P) \cap G \subset \partial G$. Since $[F_1]_p \cap [F_2]_p \subset [G]_p$ contains points in depth 2 a contradiction is easily deduced. Namely, take $p' \in F_1 \cap F_2 \cap S^2(P)$, a regular neighborhood $V' \subset P$ of p and set $V = G \cap V'$. Then $V \backslash S^2(P) \subset S^1(P)$ is disconnected containing two non-empty components which lie in F_1, F_2, respectively. However, $\mathrm{Int}(V) = S^1(P) \cap V$ and $\partial V = S^2(P) \cap V$ when V is considered as an open submanifold of G. This gives a contradiction because such neighborhoods where the boundary separates the interior do not exist in a manifold with corners.

The last implication (c) to (b) is left: An $F \in \mathcal{F}^1(P)$ is covered by regular charts on P centered at points of F. If (U, φ) is such a chart then we have $F \cap U \in \mathcal{F}^1(U)$ because of (c). Because a 1-face of a regular chart is in coordinates equivalent to a 1-face of $\mathbb{R}_+^{\dim(P)}$ the Theorem 3 applies. $\square$

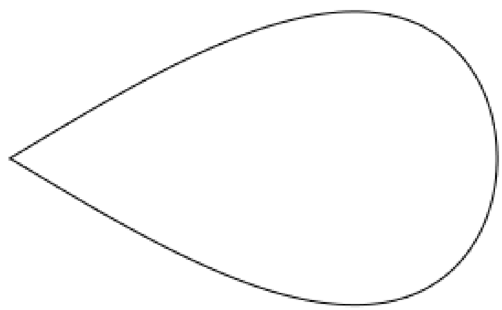

Figure 3: The teardrop $T = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y^2 \leq x^2 - x^4\}$ is an example taken from [8] which has a unique 1-face equal to the whole boundary. It is NOT a manifold with embedded faces. We can argue, for example, that the apex is in depth 2 but is contained in only one 1-face.

EXAMPLE 5. (i) Open subspaces of $\mathbb{R}^q_+$, regular charts on a manifold with corners, and simplexes are manifolds with embedded faces.

(ii) Product $P \times Q$ of manifolds with embedded faces is a manifold with embedded faces as well.

We will use the following lemmas for manifolds with corners to prove the next theorem about neatly embedded submanifolds of manifolds with embedded faces.

LEMMA 15 (Intersection of a Face with a Stratum of a Neat Submanifold). *Let P be a manifold with corners and $Q \subset P$ a neatly embedded submanifold with corners. Let $S \in \mathcal{S}^k(Q)$ and $F \in \mathcal{F}^l(P)$ for some $k, l \in \mathbb{N}_0$. If $F \cap S \neq \emptyset$ then $k + \operatorname{codim}_P(Q) \geq l$ and $S \subset F$.*

Proof. It suffices to prove $F \cap S \subset S$ is open in S similarly as in the proof of Lemma 6. First note that the notion of a set-germ for subsets of Q is the same with respect to Q and with respect to P because Q is topologically embedded into P. Let $p \in F \cap S$. Then $[S]_p \in \mathcal{F}^k_p(Q)$ is the highest codimension local face of Q at p; thus, it holds $[S]_p = \bigcap_{i=1}^k [F_i]_p$ where $\mathcal{F}^1_p(Q) = \{[F_1]_p, \ldots, [F_k]_p\}$, $k = d_Q(p)$. By the neatness condition (iii) of Definition 15, we may write $\mathcal{F}^1_p(P) = \{[G_1]_p \ldots, [G_k]_p, \ldots [G_{d_P(p)}]_p\}$ with $[F_i]_p \subset [G_i]_p$ for all $1 \leq i \leq k$ and $[Q]_p \subset [G_i]_p$ for all $k < i \leq d_P(p)$. In particular, every local 1-face of P at p contain at least one local 1-face of Q; hence, $[S]_p \subset [G_i]_p$ for all $i = 1, \ldots, d_P(p)$. Since $[F]_p$ is a union of local l-faces of P at p which are intersections of local 1-faces of P at p, schematically $[F]_p = \bigcup \bigcap^l [G_i]$, we see that $[F]_p \supset [S]_p$. Thereofre, $F \cap S \subset S$ is open. It is also closed and non-empty by assumption; hence, $S \subset F$. $\square$

LEMMA 16 (Intersection of a Face with a Face of a Neat Submanifold). *Let P be a manifold with corners and $Q \subset P$ a neatly embedded submanifold with* *corners. Let $G \in \mathcal{F}^k(Q)$ and $F \in \mathcal{F}^l(P)$ for some $k, l \in \mathbb{N}_0$. If there is a $p \in Q$ such that $[G]_p \subset [F]_p$ then $k + \operatorname{codim}_P(Q) \geq l$ and $G \subset F$.*

Proof. The fact that $[G]_p \subset [F]_p$ means by definition that there exists an open neighborhood $U \subset P$ of p such that $U \cap G \subset F$. If $S \in \mathcal{S}^l(Q)$ is such that $G = \overline{S}$ then $S \cap F \neq \emptyset$ and the previous lemma applies. $\square$

THEOREM 6 (Neatly Embedded Submanifold of a Manifold with Embedded Faces). *Let P be a manifold with embedded faces and $Q \subset P$ a neatly embedded submanifold with corners. Then Q itself is a manifold with embedded faces.*

Proof. In order to get a contradiction, suppose there is a $p \in Q$ such that condition (c) of Theorem 5 does not hold. Similarly as in the proof of that theorem this gives an $H \in \mathcal{F}^1(Q)$ and distinct $[F]_p, [F']_p \in \mathcal{F}^1_p(Q)$ such that $[F]_p, [F']_p \subset [H]_p$. Neatness of the embedding and neatness of P imply there are distinct $G, G' \in \mathcal{F}^1(P)$ such that $[F]_p \subset [G]_p, [F']_p \subset [G']_p$. A neat immersion does not map any two local 1-faces into the same local 1-face as may be seen from its representation as linear injection and is implied by condition (ii) of the definition; hence, neither $[F]_p \subset [G']_p$ nor $[F']_p \subset [G]_p$ hold. However, we have $[G]_p \cap [H]_p \supset [F]_p$ which implies $G \cap S = G \cap H \cap S^1(Q) \neq \emptyset$ where $S \in \mathcal{S}^1(Q)$ such that $\overline{S} = H$. Lemma 15 then implies $S \subset G$; thus, $H \subset G$. Then $[F']_p \subset [H]_p \subset [G]_p$ which is a contradiction and the lemma is proven. $\square$

REMARK 12 (Transverse Intersection of Manifolds with Embedded Faces). The previous theorem asserts images of neat embeddings and level sets of neat submersions are manifolds with embedded faces if they are contained in manifolds with embedded faces. In particular, *transverse intersection of manifolds with embedded faces is a manifold with embedded faces again.*

THEOREM 7 (Structure of the System of Faces of a Manifold with Embedded Faces). *Let P be a manifold with embedded faces and $F \in \mathcal{F}^k(P)$ for some $k \in \mathbb{N}_0$. Then F is a manifold with embedded faces neatly embedded in P,* $\operatorname{codim}_P(F) = k$*,* $d_F(p) = d_P(p) - k$ *for all* $p \in F$*, and*

$$\mathcal{F}^l(F) = \{G \in \mathcal{F}^{k+l}(P) : G \subset F\} \tag{5}$$

for every $l \in \mathbb{N}_0$. If $F_1, \ldots, F_n \in \mathcal{F}^1(P)$ for some $n \in \mathbb{N}$ then it holds

$$F_1 \cap \ldots \cap F_n = \bigsqcup \{H \in \mathcal{F}^n(P) : H \subset F_1 \cap \ldots F_n\} \tag{6}$$

as an equality of topological spaces.

On the other hand, for every $H \in \mathcal{F}^n(P)$ there are n-unique 1-faces $F_1, \ldots, F_n$ such that H is a connected component of $F_1 \cap \ldots \cap F_n$.

Proof. We will prove everything except for Equation 6 and the last paragraph by induction on $\dim(P)$:

If $\dim(P) = 0$ then P is a countable union of points and there is no face; hence, nothing to prove. Let $\dim(P) > 0$ and assume the theorem holds for all smaller dimensions. Now we will use induction on k. If $k = 0$ then F is a connected component of P and the theorem holds trivially. Let $k > 0$ and assume the theorem holds for all smaller codimensions:

Let $F \in \mathcal{F}^k(P)$. If $p \in F$ then there is a local $(k-1)$-face $[G']_p$ such that $[F]_p \subset [G']_p$. There is also a $G \in \mathcal{F}^k(P)$ such that $[F]_p \subset [G']_p \subset [G]_p$. Lemma 7 gives $F \subset G$. By the inductive assumption for smaller k, G is a manifold with embedded faces which neatly embeds in P, $\operatorname{codim}_P(G) = k-1$, $d_G(p) = d_P(p) - k + 1$ for all $p \in G$, and $\mathcal{F}^i(G) = \{H \in \mathcal{F}^i : H \subset G\}$ for any $i \in \mathbb{N}_0$. In particular, $F \in \mathcal{F}^1(G)$. By the inductive assumption on smaller dimension, F is a manifold with embedded faces which neatly embeds in G, $\operatorname{codim}_G(F) = 1$, $d_F(p) = d_G(p) - 1$ for all $p \in F$, and $\mathcal{F}^l(F) = \{E \in$

$\mathcal{F}^{l+1}(G) : E \subset F\}$ for any $l \in \mathbb{N}_0$. Rewriting these notions in terms of the corresponding notions for P we get the desired.

Let $p \in F_1 \cap \ldots \cap F_n$. Because $[F_i]_p$ are local 1-faces, there is a unique local n-face $[H']_p$ such that $[H']_p = \bigcap_{i=1}^n [F_i]_p$. There is an $H \in \mathcal{F}^k(P)$ such that $[H']_p \subset [H]_p$; hence, $H \subset F_1 \cap \ldots \cap F_n$ by repeated usage of Lemma 7. If $\tilde{H} \in \mathcal{F}^n(P)$, $\tilde{H} \subset F_1 \cap \ldots \cap F_n$ was an other k-face such that $p \in \tilde{H}$, then $[\tilde{H}]_p \subset \bigcap_{i=1}^n [F_i]_p = [H']_p \subset [H]_p$ and Lemma 7 implies $H' = H$. Therefore, the Equation 6 holds as sets. Because n-faces of P are closed and P is normal as it is a topological manifold with boundary, disjoint n-faces are separated. Therefore, the equality holds also for topological spaces.

Existence of $F_1, \ldots, F_n$ in the last paragraph follows easily from the situation on $\mathcal{F}_p(P)$. The rest follows from what was proven previously. $\square$

Remark 13 (Lattice of Faces at a Point). We see that for a manifold with embedded faces there is a one-to-one correspondence

$$\mathcal{F}_p(P) \simeq \tilde{\mathcal{F}}_p(P)$$

which holds also as an isomorphism of lattices when we take the connected component containing p as the intersection of two faces from $\tilde{\mathcal{F}}_p(P)$.

Remark 14 (Faces of Transverse Intersection). If $H \in \mathcal{F}^k(P \times_M Q)$ it follows from remark 11 that there are $F \in \mathcal{F}^l(P)$, $G \in \mathcal{F}^m(Q)$, $l+m = k$ such that $H \subset F \times_M G$. Because $P \times_M Q \subset P \times Q$ is a closed subspace we see $H \subset F \times_M G$ is closed as well. We will use the Inverse Function Theorem 2 to show that H is open in $F \times_M G$; hence, it equals a connected component: First of all, $\dim(H) = \dim(F \times_M G)$. Secondly, since the embedding of the transverse intersection is depth preserving we may write $d_H(p) = d_{P \times_M Q}(p) - k = d_{P \times Q}(p) - k = d_{F \times G}(p) = d_{F \times_M G}(p)$ and we see $\partial H \subset \partial(F \times_M G)$. Thirdly, because $H \hookrightarrow P \times Q$ and $F \times_M G \hookrightarrow P \times Q$ are immersions also $H \hookrightarrow F \times_M G$

is an immersion. Now the IFT can be indeed applied. Therefore, $H \subset F \times_M G$ is a connected component. In particular, the following equality of manifolds holds

$$\tilde{\partial}(P \times_M Q) = \tilde{\partial}P \times_M Q \sqcup P \times \tilde{\partial}Q$$

2.7 Embeddings in Euclidean Space

In this section we will prove the Euclidean embedding theorem and the theorem about approximation of a map from a manifold with corners to a manifold without boundary by a smooth map. A smooth partition of unity will be used which can be constructed exactly in the same way as for boundaryless manifolds. References can be found in [5], [4], or [11] to name a few.

DEFINITION 19 (Regular Cover). *Let P be a manifold with corners. An atlas $\mathcal{U}$ of charts on P is called a* ***regular cover of*** *P if the following holds.*

(a) *Every $(U, \varphi) \in \mathcal{U}$ is a regular chart with $\varphi(U) = B_3^+(a) \subset \mathbb{R}_+^{\dim(P)}$. We also denote $U' = \varphi^{-1}(B_2^+(a))$, $U'' = \varphi^{-1}(B_1^+(a))$. Here by $B_r(a)$ we mean an Euclidean ball centered at a with radius r. We set $B_r^+(a) = B_r(a) \cap \mathrm{R}_+^{\dim(P)}$.*

(b) *$\{U'' : (U, \varphi) \in \mathcal{U}\}$ covers P*

(c) *$\mathcal{U}$ is countable and locally finite*

DEFINITION 20 (Associated Partition of Unity). *Let P be a manifold with corners and $\mathcal{U}$ a regular cover of P. Suppose for every $(U, \varphi) \in \mathcal{U}$ there is a smooth function $\lambda : P \to [0, 1]$ and denote the set of these functions Λ. Then Λ is called* ***a partition of unity associated to $\mathcal{U}$*** *if*

(a) $\mathrm{supp}(\lambda) = \overline{U'}$ *for every $(U, \varphi) \in \mathcal{U}$ and the associated $\lambda \in \Lambda$.*

(b) *$\sum_{\lambda \in \Lambda} \lambda(p) = 1$ for every $p \in P$*

CONVENTION 3. We often denote $\mathcal{U} = \{(U_n, \varphi_n) : n \in \mathbb{N}\}$ and $\Lambda = \{\lambda_n : n \in \mathbb{N}\}$ where λ_n is associated to (U_n, φ_n) regardless whether $\mathcal{U}$ is finite or infinite. If $|\mathcal{U}| = N \in \mathbb{N}$ we implicitly mean $n \leq N$ although we write $n \in \mathbb{N}$. Also the sum in the definition is meant to be finite, from 1 to N, although we write from 1 to ∞.

Lemma 17 (Existence of Associated Partition of Unity). *Let P be a manifold with corners and $\mathcal{U} = \{(U_n, \varphi_n) : n \in \mathbb{N}\}$ a regular cover of P. Then there exists a partition of unity associated to $\mathcal{U}$.*

Proof. We define functions $\tilde{\lambda}_n : P \to [0,1]$ for every $n \in \mathbb{N}$ by the formula

$$\tilde{\lambda}_n(p) = \begin{cases} H \circ \varphi(p) & \text{for } p \in U_n \\ 0 & \text{otherwise} \end{cases}$$

where $H : B_3(a_n) \subset \mathbb{R}^{\dim(P)} \to [0,1]$ is a smooth bump function with $H \equiv 1$ on $\overline{B}_1(a_n)$, $0 < H(x) < 1$ for all $x \in B_2(a_n) \backslash \overline{B}_1(a_n)$ and $H \equiv 0$ on $B_3(a_n) \backslash B_2(a_n)$. Such function is constructed in [4, p. 51, Lemma 2.22]. Functions $\tilde{\lambda}_n$ are smooth since $\tilde{\lambda}_n$ vanishes on the open subspace $U_n \backslash \overline{U}'_n$. Functions $\lambda_n : P \to [0,1]$ are defined by the formula

$$\lambda_n(p) = \frac{\tilde{\lambda}_n(p)}{\sum_{i=0}^{\infty} \tilde{\lambda}_i(p)} \qquad \text{for every } p \in P$$

The sum in the denominator is locally a finite sum of smooth functions since $\{U_n : n \in \mathbb{N}\}$ is locally finite. It is non-zero since $\{U''_n : n \in \mathbb{N}\}$ covers P so that for every p there is an n such that $\tilde{\lambda}_n(p) = 1$. Therefore, $\Lambda = \{\lambda_n : n \in \mathbb{N}\}$ are well defined smooth function which satisfy the required properties. □

The following lemma which constructs an exhaustive system of compact sets on P is again the same as for boundary-less manifolds. It is used to cover the non-compact case in the theorem below and also in several approximation results. However, we will not deal with non-compact manifolds anymore so we put it here just for completeness.

Lemma 18 (Exhaustion by Compact Subspaces). *Let P be a manifold with corners. Then there is an open cover $\{P_n : n \in \mathbb{N}\}$ of P such that $\overline{P_n}$ is compact and $\overline{P_n} \subset P_{n+1}$ for every $n \in \mathbb{N}$.*

Proof. By Lemma 4 there is a countable basis of pre-compact sets $\{U_i : i \in \mathbb{N}\}$. It suffices to set $P_n = \bigcup_{i=1}^{n} \overline{U_i}$. $\square$

THEOREM 8 (Refininig by Regular Cover). *Let P be a manifold with corners and $\mathcal{W}$ an open cover of P. Then there is a regular cover $\mathcal{U}$ refining $\mathcal{W}$.*

Proof. If K is compact and $V \subset P$ open such that $K \subset V \subset P$ then we can construct a finite regular cover of K in V refining $\mathcal{W}$ as follows: For every $p \in K$ there is a $W \in \mathcal{W}$ with $p \in W$. We take a regular coordinate 3-ball centered at p and contained in $V \cap W$. Because K is compact there is a finite number of these 3-balls such that the corresponding 1-balls cover K. It is easy to see this is the desired finite regular cover of K refining W.

If P is compact then the construction from the preceding paragraph applies with $K = V = P$. In the non-compact case we use lemma 18 to get pre-compact $\{P_i : i \in \mathbb{N}\}$ exhausting P. We also set $P_0 = \emptyset$. For every $n \in \mathbb{N}$ we apply the construction from the first paragraph to $K_n = \overline{P_{n+1}} \backslash P_n$ and $V_n = P_{n+2} \backslash \overline{P_{n-1}}$ and get regular charts $\{(U_i^n, \varphi_i^n) : i = 1, \ldots, k_n\}$ covering K_n which are contained in V_n. Since $\{K_n : n \in \mathbb{N}\}$ covers P, the union of these charts over $n \in \mathbb{N}$ covers P as well. Because $V_n \cap V_m = \emptyset$ whenever $|n - m| > 2$ this set of charts is locally finite. It is a refinement of $\mathcal{W}$ by construction. If $\mathcal{U}$ has finite number of element then duplicate to get an U_n for every $n \in \mathbb{N}$. $\square$

LEMMA 19 (Bump Function). *Let P be a manifold with corners, $K \subset P$ a closed subspace and $U \subset P$ an open neighborhood of K. Then there is a smooth function $f : P \to [0, 1]$ such that $f \equiv 1$ on K and* $\mathrm{supp}(f) \subset U$.

Proof. Consider the cover $\mathcal{W} = \{U, P \backslash K\}$. Let $\mathcal{U} = \{(U_n, \varphi_n) : n \in \mathbb{N}\}$ be a regular refinement and λ_n the associated partition of unity. For every $p \in P$

set

$$\psi(p) = \sum_{\{n\in\mathbb{N}:U_n\subset U\}} \lambda_n(p) \qquad \text{and} \qquad \psi'(p) = \sum_{\{n\in\mathbb{N}:U_n\subset P\backslash K\}} \lambda_n(p).$$

Then $\psi(p) + \psi'(p) = \sum_{n=1}^{\infty} \lambda_n(p) = 1$ while $\mathrm{supp}(\psi) \subset U$ and $\mathrm{supp}(\psi') \subset P\backslash K$. It follows ψ is the desired map f. $\square$

The following theorem is a straightforward generalization of the simplest embedding theorem of compact manifolds into $\mathbb{R}^N$. I believe, however, that weak Whitney theorems might be proven for all manifolds with corners exactly in the same way as they are proven for manifolds without boundary. This is because they are based on taking local approximations and gluing them together by partition of unity which works the same. The following theorem shows also that a compact manifold with embedded faces neatly embeds in $\mathbb{R}^q_+$. I am not sure how to generalize this to the non-compact case.

Theorem 9 (Compact Embeddings in $\mathbb{R}^N_+$). *Let P be a compact manifold with corners. Then there is an $N \in \mathbb{N}_0$ and a smooth embedding $f : P \to \mathbb{R}^N_+$. If P is a compact manifold with embedded faces then there is an $N' \in \mathbb{N}_0$ and a neat embedding $f' : P \to \mathbb{R}^{N'}_+$ such that $d_P(p) = d_{\mathbb{R}^{N'}_+}(f'(p))$ for all $p \in P$.*

Proof. Let $\{(U_n, \varphi_n) : n = 1, \ldots, k\}$ be a finite regular cover of P and $\lambda_n : P \to [0, 1]$ the associated partition of unity. For every $n = 1, \ldots, k$ and $p \in P$ we set

$$(\lambda_n\varphi_n)(p) = \begin{cases} \lambda_n(p)\varphi_n(p) & p \in U_n \\ 0 & \text{otherwise} \end{cases}$$

This defines a smooth function $\lambda_n\varphi_n : P \to \mathbb{R}^{\dim(P)}_+$ since $(\lambda_n\varphi_n)(p) \equiv 0$ on the open subspace $U_n\backslash\overline{U'_n}$. We set

$$f = (\lambda_1\varphi_1, \ldots, \lambda_n\varphi_n, \lambda_1, \ldots, \lambda_n)$$

which is a smooth function $f : P \to \mathbb{R}^N_+$ where $N = k(\dim(P) + 1)$. We will check this is an embedding:

Let $p \in P$ and pick an $i \in \{1, \ldots, k\}$ such that $\lambda_i(p) \neq 0$. This implies $p \in U'_i$. In order to prove injectivity of f suppose, for a contradiction, there is a $p' \in P$ such that $f(p) = f(p')$. It follows that $p' \in U_i$ since $\lambda_i(p') = \lambda_i(p)$. Therefore, $\varphi_i(p) = \varphi_i(p')$ which implies $p = p'$ since φ_i is injective. As a result, f is injective. Now we will check it is an immersion. Let $v \in T_pP$. Then either $(\mathrm{d}\lambda_i)_p(v) \neq 0$ which implies $\mathrm{d}f_p(v) \neq 0$, or $(\mathrm{d}\lambda_i)_p(v) = 0$. In the latter case, $\mathrm{d}(\lambda_i\varphi_i)_p(v) = \lambda_i(p)(\mathrm{d}\varphi_i)_p(v)$ which is non-zero since φ_i is an immersion. Therefore, $f : P \to \mathbb{R}^N_+$ is a smooth injective immersion. If $K \subset P$ is closed it is automatically compact since P is compact; hence, $f(K) \subset \mathbb{R}^N_+$ is compact as well. However, a compact set $f(K)$ in $\mathbb{R}^N_+$ is closed; hence, f is an open map. It follows f is a smooth embedding.

Let P be a manifold with embedded faces and let $f : P \to \mathbb{R}^N_+$ be the smooth embedding constructed above. There is a finite number of faces of P because P is covered by a finite number of regular charts which have finite number of faces. Therefore, we may assume $\mathcal{F}^1(P) = \{F_1, \ldots, F_n\}$. For every $i = 1, \ldots, n$ we will need a smooth function $h_i : P \to [0, 1]$ such that $F_i = h_i^{-1}(0)$ and $\mathrm{Ker}((\mathrm{d}h_i)_p) = T_pF$ for every $p \in F$:

Such functions can be constructed either directly or by referring to an independent result in an upcoming section 2.10; namely, to the collar neighborhood Theorem 17. It asserts that for every $F \in \mathcal{F}^1(P)$ there is an open neighborhood $C(F) \subset P$ of F and a diffeomorphism $\varphi : F \times [0, 1) \to C(F)$ such that $\varphi(p, 0) = p$ for all $p \in F$. We pick an $F \in \mathcal{F}^1(P)$ and denote $\tilde{h} = (\varphi^{-1})^2 : C(F) \to [0, 1]$. It clearly holds $F = \tilde{h}^{-1}(0)$ and we will check the second condition. The inclusion $T_pF \subset \mathrm{Ker}(\mathrm{d}\tilde{h}_p)$ for $p \in F$ is clear as F is a level set of $\tilde{h}$. For the other inclusion, pick a non-zero element

$1 \in T_0[0,1] \simeq \mathbb{R}$. We have $\mathrm{d}\varphi_{(p,0)}(T_pF \oplus 0) = T_pF$ as $\varphi(p,0) = p$ for $p \in F$ and from dimensional reasons. Therefore, $X_p = \mathrm{d}\varphi_{(p,0)}(0 \oplus 1) \notin T_pF$ is a generator of 1-dimensional T_pP/T_pF since $\mathrm{d}\varphi_{(p,0)}$ is an isomorphism. Then we may calculate for every $p \in F$ using the chain rule

$$1 = \mathrm{d}(\mathrm{id}_{[0,1]})_0(1) = (\mathrm{d}(\tilde{h} \circ \varphi))_{(p,0)}(0 \oplus 1) = (\mathrm{d}\tilde{h})_p(\mathrm{d}\varphi)_{(p,0)}(0 \oplus 1) = (\mathrm{d}\tilde{h})_p X_p.$$

If $v \notin T_pF$ we may write $v = w + \lambda X_p$ for some $w \in T_pF$ and $\lambda \neq 0$. We see that $(\mathrm{d}\tilde{h})_p(v) = \lambda \neq 0$. Therefore, $\mathrm{Ker}(\mathrm{d}\tilde{h}_p) \subset T_pF$ and the equality follows. The map $\tilde{h} : C(F) \to [0,1]$ satisfies both requirements. Nevertheless, its domain is only $C(F)$ and not P so we need to extend it. Referring to Lemma 19, there is a smooth function $\psi : P \to [0,1]$ supported in $C(F)$ with $\psi \equiv 1$ on $K = \varphi\left(F \times [0, \frac{1}{2}]\right)$ which is a closed subspace of P as it is compact. We define

$$h(p) = (1 - \psi(p)) + (\psi\tilde{h})(p)$$

for all $p \in P$ where the smooth function $\psi\tilde{h}$ is defined similarly as in the first paragraph. If $p \in F$ then clearly $h(p) = 0$. If $h(p) = 0$ then $\psi(p) = 1$; hence, $p \in C(F)$; and $0 = (\psi\tilde{h})(p) = \tilde{h}(p)$ which implies $p \in F$. Therefore, $h^{-1}(0) = F$ as we wanted. Because h equals $\tilde{h}$ on a neighborhood K of F it follows $\mathrm{Ker}((\mathrm{d}h)_p) = \mathrm{Ker}((\mathrm{d}\tilde{h})_p) = T_pF$ for all $p \in F$. Finally, we see that $h : P \to [0,1]$ is the function with $F = h^{-1}(0)$ and $\mathrm{Ker}((\mathrm{d}h)_p) = T_pF$ for $p \in F$ which is what we wanted.

Having constructed the functions h_i for every $i = 1, \ldots, n$ we set $N' = N + n$ and define a smooth function $f' : P \to \mathbb{R}_+^{N'}$ by the formula

$$f'(p) = (f_1(p) + 1, \ldots, f_N(p) + 1, h_1(p), \ldots, h_n(p))$$

for every $p \in P$. We write $f' = (f, h)$. It is an injective immersion because f is. Therefore, it is a smooth embedding of compact P. It remains to check conditions (ii) and (iii) of definition of a neat immersion 15. By construction it holds $p \in F_i$ if and only if $f'(p) \in \partial_{N+i}\mathbb{R}^{N'}_+$. Moreover, $I_{f'(p)} \subset \{N+1, \ldots, N'\}$ as $f > 0$. Therefore, if $\mathcal{F}_p(P) = \{[F_{i_1}]_p, \ldots, [F_{i_k}]_p\}$ then $\mathcal{F}_{f'(p)}(\mathbb{R}^{N'}_+) = \{\partial_{i_1}\mathbb{R}^{N'}_+, \ldots, \partial_{i_k}\mathbb{R}^{N'}_+\}$ and $[F_{i_l}]_p \subset f'^*[\partial_{i_l}\mathbb{R}^{N'}_+]_{f'(p)}$; thus, condition (iii) is satisfied. In addition, $d_P(p) = d_{\mathbb{R}^{N'}_+}(f'(p))$ for all $p \in P$. As for the second condition, we need to prove that the induced map $T_pP/C_pP \to T_{f'(p)}\mathbb{R}^{N'}_+/C_{f'(p)}\mathbb{R}^{N'}_+$ is injective for all $p \in P$. If $p \in \mathrm{Int}(P)$ then $f'(p) \in \mathrm{Int}(\mathbb{R}^{N'}_+)$ and $C_pP = T_pP$, $C_{f'(p)}\mathbb{R}^{N'}_+ = T_{f'(p)}\mathbb{R}^{N'}_+$; thus, there is nothing to prove. To cover the other cases we may assume, without loss of generality, that $\mathcal{F}^1_p(P) = \{F_1, \ldots, F_k\}$ for some $1 \leq k \leq n$. Then $C_pP = \bigcap_{i=1}^k T_pF_i$ and $C_{f'(p)} = \{v \in \mathbb{R}^{N'} : v^i = 0 \text{ for all } i \leq k\}$. It suffices to show that if $(\mathrm{d}f')_p(v) \in C_{f'(p)}$ for some $v \in T_pP$ then $v \in C_pP$. However, $(\mathrm{d}f')_p(v) \in C_{f'(p)}$ implies $(\mathrm{d}h_i)_p(v) = 0$ for all $1 \leq i \leq k$ which indeed implies $v \in C_pP$ since $\mathrm{Ker}((\mathrm{d}h_i)_p) = T_pF_i$. Therefore, f' is a neat embedding and the theorem is proven. $\square$

Example 6. (i) Note that a general manifold with corners can not be neatly embedded in $\mathbb{R}^N_+$. This is dictated by Lemma 6 which tells us that only manifolds with embedded faces may neatly embed into manifolds with embedded faces. The teardrop from Remark 2.6 is an example.

(ii) The defining embedding in $\mathbb{R}^{q+1}_+$ of the standard simplex Δ^q is a neat embedding. However, Δ^q can not be neatly embedded in $\mathbb{R}^q_+$ as it has more than one q-face.

Theorem 10 (Approximation by a Smooth Map). *Let P, M be manifolds with corners, $\partial M = \emptyset$ and let $K \subset P$ be a closed subset. Let $f : P \to M$ be a continuous function which is smooth on K. Then there is a smooth function $\tilde{f} : P \to M$ which is homotopic to f relative to K.*

Proof. Identify M with its image under an embedding $M \hookrightarrow \mathbb{R}^N$. This may be done because of an embedding theorem for manifolds without boundary in [4, p. 251, Theorem 10.11]. There is an open neigbourhood $U \subset \mathbb{R}^N$ of M and a smooth submersion $r : U \to M$ such that $r(m) = m$ for all $m \in M$. This is a consequence of the tubular neighborhood theorem for manifolds without boundary in [4, p. 255, Theorem 10.19]. From the same source, there is a smooth function $\varepsilon : M \to (0,1)$ such that $B_{\varepsilon(m)}(m) \subset U$ for every $m \in M$. In the rest of the proof we will construct a smooth function $g : P \to \mathbb{R}^N$ such that $|f(p) - g(p)| < \varepsilon(f(p))$ for all $p \in P$ and $g \equiv f$ on K. Then it suffices to set $\tilde{f} = r \circ g$ to get the desired smooth function. It is homotopic to f relative to K through the homotopy $H : P \times [0,1] \to M$ defined for all $(p,t) \in P \times [0,1]$ by the formula

$$H(p,t) = r((1-t)f(p) + tg(p)).$$

Now we will construct g: For every $p \in P \backslash K$ we choose an open neighborhood $W_p \subset P \backslash K$ of p such that there is an $\varepsilon_p > 0$ with $\varepsilon(p') > \varepsilon_p$ and $\|f(p') - f(p)\| < \varepsilon_p$ for every $p' \in W_p$. This is possible since ε and f are continuous. We define a constant function $g_p : W_p \to \mathbb{R}^N$ as $g_p(p') = f(p)$ for every $p' \in W_p$. For every $p \in K$ we choose an open neighborhood $W_p \subset P$ of p such that there is a smooth extension $g_p : W_p \to \mathbb{R}^N$ with $g_p\big|_{K \cap W_p} = f\big|_{K \cap W_p}$ and an $\varepsilon_p > 0$ with $\varepsilon(p') > \varepsilon_p$, $\|g_p(p') - f(p')\| < \varepsilon_p$ for every $p' \in W_p$. This is possible by continuity of the maps again.

Let $\mathcal{U} = \{(U_n, \varphi_n) : n \in \mathbb{N}\}$ be a regular cover refining the cover $\mathcal{W} = \{W_p : p \in P\}$ and let $\{\lambda_n : U_n \to [0,1]$ for $n \in \mathbb{N}\}$ be the associated partition of unity. Let $p_n \in P$ be such that $U_n \subset W_{p_n}$ for every $n \in \mathbb{N}$. Define smooth functions $\lambda_n g_{p_n} : P \to \mathbb{R}^N$ by $(\lambda_n g_{p_n})(p') = \lambda_n(p') g_{p_n}(p')$ if $p' \in U_n$ and $\lambda_n g_{p_n}(p') = 0$ if $p' \in P \backslash \overline{U_n'}$. For every $p \in P$ define $g(p) = \sum_{n=1}^{\infty} \lambda_n g_{p_n}(p)$. It is well defined and smooth since it locally equals a finite sum of smooth functions. If $p \in K$ then $g_{p_n}(p) = f(p)$ for every $n \in \mathbb{N}$ such that $p \in U_n$; therefore, $g(p) = f(p)$ because $\sum_{n=1}^{\infty} \lambda_n(p) = 1$. For every $p \in P$ we have

$$\|g(p) - f(p)\| = \|\sum_{n=1}^{\infty} \lambda_n g_n(p) - \sum_{n=1}^{\infty} \lambda_n(p) f(p)\| \leq \sum_{n=1}^{\infty} \lambda_n(p)\|g_n(p) - f(p)\| \leq$$

$$\leq \sum_{n=1}^{\infty} \lambda_n(p)\varepsilon_{p_n} < \sum_{n=1}^{\infty} \lambda_n(p)\varepsilon(p) = \varepsilon(p)$$

Therefore, g is the required smooth function approximating f. $\square$

This theorem might be generalized to the case $\partial M \neq \emptyset$ as follows. First, we neatly embed M in $\mathbb{R}_+^N$. We take a special normal bundle of M in $\mathbb{R}_+^N$ whose vectors point along the faces of $\mathbb{R}_+^N$. Then we construct a tubular neighborhood U of M in $\mathbb{R}_+^N$ similarly as in the boundary-less case. Except for the construction of a tubular neighborhood, the rest would be the same as above.

2.8 Transverse Approximation

In this section we will show that any map from a manifold with corners to a manifold without boundary can be homotoped to a map transverse to a given map. The proofs are almost identical to the proofs in the boundary-less case which can be found in [11], for example.

THEOREM 11 (Parametric Transversality Theorem). *Let P, Q, B, M be manifolds with corners such that $\partial M = \emptyset = \partial B$. Let $F : P \times B \to M$ and $g : Q \to M$ be smooth maps such that $F \pitchfork g$. Let $F_b : P \to M$ be defined by $F_b(p) = F(p, b)$ for all $p \in P$ and $b \in B$. Then the subspace $\{b \in B : F_b \pitchfork g\}$ is dense in B.*

Proof. The proof relies on showing that the set of $b \in B$ such that F_b is not transverse to g is of measure zero in B. The complement of this set is then dense in B. See [4, p. 242-246] for the definition of measure zero subsets of manifolds without boundary and their basic properties. According to Lemma 13, it holds $F_b \pitchfork g$ if and only if $S^k F_b \pitchfork S^l g$ for all $k, l \in \mathbb{N}_0$. Therefore, we can write

$$\{b \in B : F_b \text{ not tranverse to } g\} = \bigcup_{k,l \in \mathbb{N}} \{b \in B : S^k F_b \text{ not transverse to } S^l g\}$$

Since a countable union of measure zero subsets is of measure zero it suffices to prove that for each $k, l \in \mathbb{N}$ the set $\{b \in B : S^k F_b \pitchfork S^l g\}$ has measure zero. We see that the problem is completely reduced to transverse intersection of manifolds without boundary. However, the proof that $\{b \in B : F_b \cap g\}$ has measure zero in B when $\partial P = \emptyset = \partial Q$ uses Sard's Theorem and might be found in [7, p. 68, The Transversality Theorem]. $\square$

LEMMA 20 (Extension to a Submersion). *Let P, M be manifolds with corners, $\partial M = \emptyset$, and let $f : P \to M$ be a smooth map. Then there is a smooth* *map $F : P \times B \to M$ where $B \subset \mathbb{R}^{\dim(M)}$ is the unit ball centered at 0 such that $F\big|_{\{p\}\times B} : B \to M$ is a submersion and $F(p,0) = f(p)$ for all $p \in P$.*

Proof. Similarly as in the proof of Theorem 10 we may assume $M \subset \mathbb{R}^N$ is an embedded submanifold. There is an open neighborhood $U \subset \mathbb{R}^N$ of M, a smooth retraction $r : U \to M$, and there is a smooth function $\varepsilon : M \to U$ such that $B_{\varepsilon(m)}(m) \subset U$ for all $m \in M$. We define the smooth function $F : P \times B \to M$ by setting

$$F(p,b) = r(f(p) + \varepsilon(f(p))b)$$

for all $(p,b) \in P \times B$. This clearly satisfies all required properties. $\square$

THEOREM 12 (Transverse Homotopy Theorem). *Let P, Q, M be manifold with corners such that $\partial M = \emptyset$ and Q is compact. Let $f : P \to M$, $g : Q \to M$ be smooth maps. Let $K \subset P$ be a closed subset such that $f \pitchfork_K g$. Then there is a smooth homotopy $H : P \times [0,1] \to M$ relative to K with $H_0 = f$ and $H_1 \pitchfork g$.*

Proof. By Lemma 14 there is an open neighborhood $U \subset P$ of K such that $f \pitchfork_U g$. Consider the smooth map $F : P \times B \to M$ from Lemma 20. Let $\chi : P \to [0,1]$ be a smooth bump function with $\chi\big|_{P\setminus U} = 1$ and $\operatorname{supp}(\chi) \subset P\setminus K$ from Lemma 19. Assume that χ satisfies: if $\chi(p) = 0$ for some $p \in P$ then $\mathrm{d}\chi_p = 0$; otherwise, replace χ by χ^2. Set

$$\tilde{F}(p,b) = F(p,\chi(p)b)$$

for all $(p,b) \in P \times B$. We will check that $\tilde{F} \pitchfork g$: We have

$$\mathrm{d}\tilde{F}_{(p,b)}(C_pP \oplus T_bB) = \mathrm{d}\left(\tilde{F}\Big|_{P\times\{b\}}\right)_p (C_pP) + \mathrm{d}\left(\tilde{F}\Big|_{\{p\}\times B}\right)_b (T_bB) \qquad (7)$$

where $\mathrm{d}\tilde{F}\Big|_{\{p\}\times B} = \chi(p)\mathrm{d}F\big|_{\{p\}\times B}$. Let $(p,b,q) \in \{\tilde{F} = g\} \subset P \times B \times Q$, $m = g(q) = \tilde{F}(p,b)$ and distinguish two cases:

If $\chi(p) > 0$ then

$$\mathrm{d}\left(\tilde{F}\Big|_{\{p\}\times B}\right)_b (T_bB) = \chi(p)\mathrm{d}\left(F\big|_{\{p\}\times B}\right)_b (T_bB) = T_mM.$$

Plugging this into Equation 7 shows that $\mathrm{d}\tilde{F}_{(p,b)}(C_pP \oplus T_bB) = T_mM$; hence, $\mathrm{d}\tilde{F}_{(p,b)}(C_pP \oplus T_bB) + \mathrm{d}g_q(C_qQ) = T_mM$ holds trivially.

If $\chi(p) = 0$ then $p \in U$, $\mathrm{d}\chi_p = 0$ and we may calculate

$$\mathrm{d}\tilde{F}_{(p,b)}(C_pP \oplus T_bB) + \mathrm{d}g_q(C_qQ) = \mathrm{d}\left(\tilde{F}\Big|_{P\times\{b\}}\right)_p (C_pP) + \mathrm{d}g_q(C_qQ) =$$

$$= \mathrm{d}F_{(p,\chi(p)b)}(C_pP \oplus \mathrm{d}\chi_p b) + \mathrm{d}g_q(C_qQ) = \mathrm{d}f_p(C_pP) + \mathrm{d}g_q(C_qQ) = T_mM$$

where the last equality holds because $f \pitchfork_U g$. Therefore, the smooth map $\tilde{F} : P \times B \to M$ satisfies $\tilde{F} \pitchfork g$ while $\tilde{F}(p,b) = f(p)$ for $(p,b) \in K \times B \cup P \times \{0\}$. We apply Theorem 11 to $\tilde{F}$ and get a $b \in B$ such that $\tilde{F}_b \pitchfork g$. Now it suffices to set $H(p,t) = \tilde{F}(p,tb)$ for every $(p,t) \in P \times [0,1]$ in order to get the desired homotopy. $\square$

2.9 Orientation

In this section we will define orientation, boundary orientation and transverse orientation of manifolds with corners and prove a theorem about associativity and graded commutativity of the transverse intersection. The idea is to define an orientation of a manifold with corners P as an orientation of a manifold without boundary $\mathrm{Int}(P)$ and the boundary orientation of $F \in \mathcal{F}^1(P)$ as the boundary orientation of $\mathrm{Int}(F)$ in the manifold with boundary $\mathrm{Int}(P) \cup \mathrm{Int}(F)$. However, in what follows we will define the orientation explicitly.

DEFINITION 21 (Positively Oriented Charts). *Let P be a manifold with corners with $\dim(P) > 0$. Two charts (U, φ) and (U', φ') are* ***positively oriented*** *if*

$$\det(\mathrm{d}(\varphi' \circ \varphi^{-1})_x) > 0 \tag{8}$$

for every $x \in \varphi(U \cap U')$. We say P is ***orientable*** *if either $\dim(P) = 0$ or there is a cover of $\mathrm{Int}(P)$ consisting of mutually positively oriented charts.*

REMARK 15 (Orientation is Equivalent to Orientation of the Interior). Note that charts (U, φ), (U', φ') are positively oriented if and only if $(\mathrm{Int}(U), \varphi)$, $(\mathrm{Int}(U'), \varphi')$ are positively oriented: Recall that $U \cap U' \neq \emptyset$ if and only if $\mathrm{Int}(U) \cap \mathrm{Int}(U') \neq \emptyset$ for any two open subspaces of a manifold with corners. The claim follows easily by continuity of determinant.

LEMMA 21 (Two Orientations). *Let P be a connected orientable manifold with corners with $\dim(P) > 0$. Then the relation "being positively oriented" is an equivalence relation on charts and there are precisely two equivalence classes.*

Proof. Let $\mathcal{U}$ be a cover of $\mathrm{Int}(P)$ whose charts are positively oriented. Let $\mathcal{P}^+$ be the set of charts which are positively oriented with $\mathcal{U}$ and let $\mathcal{P}^-$ be the rest. For a chart (U, φ) let $S^+ \subset U$ / $S^- \subset U$ be a set consisting of those $p \in \mathrm{Int}(U)$ such that there exists a chart $(U', \varphi') \in \mathcal{U}$ at p such that the Equation 8 with $x = \varphi(p)$ holds / does not hold. Clearly, $\mathrm{Int}(U) = S^+ \cup S^-$ because $\mathcal{U}$ is a cover of $\mathrm{Int}(P)$ and $S^\pm \subset \mathrm{Int}(U)$ are open subspaces because the determinant is continuous. Moreover, $S^+ \cap S^- = \emptyset$ using the fact that charts in $\mathcal{U}$ are positively oriented and the chain rule. Therefore, either $\mathrm{Int}(U) = S^+$ or $\mathrm{Int}(U) = S^-$ since U is connected and $\mathrm{Int}(U)$ as well. It follows, using the chain rule again, that two charts are positively oriented if and only if they both lie in $\mathcal{P}^+$ or $\mathcal{P}^-$. We see that "being positively oriented" is an equivalence relation and the classes $\mathcal{P}^\pm$ do not depend on the choice of $\mathcal{U}$. $\square$

DEFINITION 22 (Orientation). *Let P be an orientable manifold with corners. We define an* ***orientation*** *on P as a choice of an equivalence class of positively oriented charts. Charts from the chosen class are called* ***positively oriented charts****. If $\dim(P) = 0$ we define the orientation as a choice of a continuous map $o : P \to \{-1, 1\}$.*

An orientable manifold with corners P together with a choice of an orientation is called an ***oriented manifold with corners****. If P is oriented we denote $-P$ the oppositely oriented manifold, i.e. such that the orientation of every connected component is reversed. We call positively oriented charts on $-P$* ***negatively oriented charts*** *on P.*

REMARK 16 (Oriented Cover). If P is oriented and $\dim(P) \geq 2$ then positively oriented charts cover P: Every point is contained in a chart which is either positively or negatively oriented. However, we can swap two coordinate and turn a negatively oriented chart into a positively oriented one.

If $\dim(P) = 1$ and $p \in \partial P$, then the determinant of any transition function ψ at p is positive as $\mathrm{d}\psi_p(T_p^+ P) \subset T_p^+ P$ by Lemma 8. Therefore, the charts containing p are all either positively or negatively oriented. As an example, $P = [0, 1]$ is orientable but does not admit an oriented atlas: If $0 \mapsto 1$ is a positive direction then $\{1\}$ is not contained in any positively oriented chart whereas $\{0\}$ is. The opposite holds for $1 \mapsto 0$.

Now we are going to define the boundary orientation induced on 1-faces of an orientable manifold with embedded faces. We adopt the convention that $\partial_i \mathbb{R}^q_+$ is always identified with $\mathbb{R}^{q-1}_+$ via the diffeomorphism

$$(x_1, \ldots, x_{q-1}) \in \mathbb{R}^{q-1}_+ \mapsto (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_{q-1}) \in \partial_i \mathbb{R}^q_+ \qquad (9)$$

for $i = 1, \ldots, q$ and $q \in \mathbb{N}$.

Lemma 22 (Existence of Boundary Orientation). *Let P be an oriented manifold with embedded faces,* $\dim(P) \geq 2$ *and let $F \in \mathcal{F}^1(P)$. Restrictions to F of positively oriented charts on P which map F into $\partial_i \mathbb{R}^{\dim(P)}_+$ for i odd / even , respectively, constitute oppositely oriented covers of* $\mathrm{Int}(F)$.

Proof. First of all, such restrictions for i odd / even cover $\mathrm{Int}(F)$: To see this, let $p \in \mathrm{Int}(F)$ and let (U, φ) be a regular chart containing p. In most of the cases it suffices to switch one or two pairs of coordinates in order to get a positively oriented chart ψ out of φ which maps F into $\partial_i \mathbb{R}^{\dim(P)}_+$ for i odd / even. The only case which requires attention is when $\dim(P) = 2$, (U, φ) is positively oriented and maps F into $\partial_2 \mathbb{R}^2_+$ / $\partial_1 \mathbb{R}^2_+$. Then the required chart ψ can be constructed from φ on a neighborhood of p as $\psi = (\varphi_2, C - \varphi_1)$ / $(C - \varphi_2, \varphi_1)$ where C is big enough. This works well since $p \in \mathrm{Int}(F)$.

Now we will check compatibility: Let (U, φ), (U', φ') be positively oriented charts on P such that φ maps F into $\partial_i \mathbb{R}^q_+$ and φ' into $\partial_{i'} \mathbb{R}^q_+$ for some $i, i' \in \mathbb{N}$. Then we have

$$\left. \frac{\partial (\varphi' \circ \varphi^{-1})^{i'}}{\partial x^k} \right|_x = 0$$

for all $k \neq i$ where $x = \varphi(p)$. For $k = i$ this derivative has to be non-zero by regularity of the transition function and positive since the inward pointing cone is preserved. Let $\tilde{U} = U \cap F$, $\tilde{\varphi} = \pi \circ \varphi$ where $\pi : \mathbb{R}_+^{\dim(P)} \to \mathbb{R}^{\dim(P)-1}$ is the projection to the plane $x^i = 0$ and similarly for (U', φ') and i'. Then we may calculate using the Determinant Expansion Formula

$$\det(\mathrm{d}(\varphi' \circ \varphi^{-1})_{\pi(x)}) = (-1)^{i+i'} \frac{\partial(\varphi' \circ \varphi^{-1})^{i'}}{\partial x^i}\bigg|_x \det(\mathrm{d}(\tilde{\varphi}' \circ \tilde{\varphi}^{-1})_x). \tag{10}$$

We see $(\tilde{U}, \tilde{\varphi})$ and $(\tilde{U}', \tilde{\varphi}')$ are positively oriented if and only if $i \equiv i' \pmod 2$ which proves the lemma. □

Definition 23 (Boundary Orientation). *Let P be an oriented manifold with embedded faces and let $F \in \mathcal{F}^1(P)$. We define the **boundary orientation of** F as follows:*

(a) If $\dim(P) \geq 2$ we equip F with the orientation induced by the cover of $\mathrm{Int}(F)$ for i odd from Lemma 22.

(b) If $\dim(P) = 1$ then we define the orientation $o : F \to \{-1, 1\}$ by setting $o(p) = \pm 1$ for $p \in F$ if there exists a positively / negatively oriented chart on P containing p.

Remark 17. (i) It is well defined when $\dim(P) = 1$ by Remark 16.

(ii) Note that it is required to make a choice i odd / i even.

(iii) It is clear that $P \mapsto -P$ implies $F \mapsto -F$.

Remark 18 (Boundary Orientation of $\mathbb{R}^q_+$). For every $q \in \mathbb{N}$ consider $P = \mathbb{R}^q_+$ oriented so that the global chart $(\mathbb{R}^q_+, \mathrm{id})$ is positively oriented. The boundary orientation on $\partial_i \mathbb{R}^q_+$ is equal to $(-1)^{i+1}$ times the orientation induced by the canonical identification 9.

Lemma 23 (Orientation on 2-faces). *Let P be an oriented manifold with embedded faces and let $G, G' \in \mathcal{F}^1(P)$ be distinct 1-faces equipped with the* *boundary orientation. If $F \in \mathcal{F}^1(G) \cap \mathcal{F}^1(G')$ then the boundary orientations induced on F from G and G' disagree.*

Proof. We may assume $\dim(P) \geq 2$ since otherwise there is no 2-face and the lemma holds trivially. Let $p \in F$ and let (U, φ) be a positively oriented chart on P at p. We may arrange, by swapping the coordinates or G and G', that φ maps G into $\{x^1 = 0\}$ and G' into $\{x^i = 0\}$ for some $i \neq 1$ which might be odd / even, respectively. By definition of the boundary orientation the restriction to G is positively oriented and the restriction to G' is positively / negatively oriented according to i being odd / even, respectively. The case $\dim(P) = 2$ follows immediately since i can be only even. If $\dim(P) > 2$ consider that φ maps F into $\{x^1 = x^i = 0\}$. This is negatively / positively oriented as a boundary of $\{x^1 = 0\}$ since $i - 1$ is even / odd and we use the standard identification 9. Therefore, F is negatively / positively oriented as a boundary of G. By the same reasoning we deduce $\{x^1 = x^i = 0\}$ is positively oriented as a boundary of $\{x^i = 0\}$. Therefore, F is positively / negatively oriented as a boundary of G'. Since in both possible cases i odd / even the orientations disagree, the lemma is proven. □

Definition 24 (Orientation Preserving Map). *Let P, Q be oriented manifolds with corners such that $\dim(P) = \dim(Q)$ and let $f : P \to Q$ be a smooth map. Then f **preserves orientation** if for every pair of positively oriented charts (U, φ) on P, (W, ψ) on Q, such that $f(U) \subset W$ and for every $x \in \varphi(U)$ holds*

$$\det(\mathrm{d}(\psi \circ f \circ \varphi^{-1})_x) > 0 \tag{11}$$

*if $\dim(P) > 0$ and $o_Q(f(p)) = o_P(p)$ for every $p \in P$ if $\dim(P) = 0$. Similarly we define f **reverses orientation** with signs flipped in the equations above.*

Lemma 24 (Induced Orientation). *Let P, Q be manifolds with corners such that* $\dim(P) = \dim(Q)$ *and let Q be oriented. Let $f : P \to Q$ be a smooth immersion. Then P is orientable and can be oriented in a unique way such that f preserves orientation.*

Proof. If $\dim(P) = 0$ then we define $o_P : P \to \{-1, 1\}$ by the formula $o_Q(f(p)) = o_P(p)$ for every $p \in P$. Therefore, we may assume $\dim(P) > 0$. Consider the set $\mathcal{U}$ of charts (U, φ) on P such that there is a positively oriented chart (W, ψ) on Q with $f(U) \subset W$ and $\det(\mathrm{d}(\psi \circ f \circ \varphi^{-1})_x) > 0$ for all $x \in \varphi(U)$. First of all, we have $f(\mathrm{Int}(P)) \subset \mathrm{Int}(Q)$ by Lemma 9 and we may easily deduce that $\mathcal{U}$ covers $\mathrm{Int}(P)$. This can be done by picking the right chart at $p \in \mathrm{Int}(P)$ within the pre-image of a positively oriented chart at $f(p)$. Secondly, let $(U, \varphi), (U', \varphi') \in \mathcal{U}$. By remark 15 it suffices to show $(\mathrm{Int}(U), \varphi)$ and $(\mathrm{Int}(U'), \varphi')$ are positively oriented. However, $f : \mathrm{Int}(P) \to \mathrm{Int}(Q)$ is locally invertible by the Inverse Function Theorem. Therefore, we may write $(\varphi' \circ \varphi^{-1}) = (\varphi' \circ f^{-1} \circ \psi^{-1}) \circ (\psi \circ f \circ \varphi^{-1}) = (\psi \circ f \circ \varphi'^{-1})^{-1} \circ (\psi \circ f \circ \varphi^{-1})$ near points $p \in \mathrm{Int}(U) \cap \mathrm{Int}(U')$ and differentiate using the chain rule. It follows easily from the assumptions that the determinant is bigger than zero. $\square$

Definition 25 (Orientation of the Tangent Space). *Let P be an oriented manifold with corners and $p \in P$. Denote $n = \dim(P)$. We define the **orientation of** T_pP as follows:*

If $n \geq 2$ we take a positively oriented chart (U, φ) at p and declare the standard basis $(e_1, \ldots, e_n)$ of $\mathbb{R}^n$ to be positively oriented basis of T_pP under the identification $T_pP \overset{\varphi}{\simeq} \mathbb{R}^n$. If $n = 1$ and the chart at p is negatively oriented we declare $-e_1$ to be the positively oriented basis of T_pP. If $n = 0$ we orient $T_pP = \{0\}$ by the function $o_{T_pP} : T_pP \to \{-1, 1\}$ with the only value $o(p)$ where $o : P \to \{-1, 1\}$ is the orientation on P.

Remark 19 (Relation to Positively Oriented Charts). Obviously, two charts are positively oriented if and only if the orientation induced on T_pP by them agree for all p in the common domain.

Definition 26 (Orientation Consistent with a SES). *Let A, B, C be oriented finite dimensional vector spaces fitting into a short exact sequence*

$$0 \to A \overset{i}{\hookrightarrow} B \overset{j}{\twoheadrightarrow} C \to 0.$$

*Then we say that A, B, C are **oriented consistently with the SES** if the following holds: If $(v_1, \dots, v_k)$ is a positively oriented basis of A, $(c_1, \dots, c_l)$ is a positively oriented basis of C and $w_i \in j^{-1}(c_i)$ for every $i = 1, \dots, l$ then*

$$(b_1, \dots, b_{l+k}) = (i(v_1), \dots, i(v_k)) \times (w_1, \dots, w_l) \tag{12}$$

is a positively oriented basis of B. If a vector space $X \in \{A, B, C\}$ is zero then its oriented basis is replaced by $o(X) \in \{-1, 1\}$ and the product of bases by multiplication with $o(X)$.

Lemma 25 (Existence of Consistent Orientation). *Let A, B, C be finite dimensional vector spaces which fit into a short exact sequence $0 \to A \overset{i}{\hookrightarrow} B \overset{j}{\twoheadrightarrow} C \to 0$. If two of the vector spaces are oriented then there is a unique orientation on the third one such that A, B, C are consistently oriented with the SES.*

Proof. If $B = 0$ then $A = 0 = C$ by exactness. Therefore, orientation of the third space is recovered from the equation $o(B) = o(A)o(C)$. If $B \neq 0$ and $A = 0$ we get the right orientations from the rule that $o(A) = 1$ if and only if $B \overset{j}{\simeq} C$ is an orientation preserving isomorphism. Similarly if $C = 0$. Let all the spaces be non-zero and let A and C be oriented. First of all, oriented bases of B determined by Formula 12 which correspond to different representants $w_i \in j^{-1}(c_i)$ are positively oriented: If $w_i' \in j^{-1}(c_i)$ then

$w_i' = w_i + i(a_i')$ for some $a_i' \in A$ by exactness. Therefore, the transition matrix $(i(a_1), \ldots, i(a_k), w_1, \ldots, w_l) \mapsto (i(a_1), \ldots, i(a_k), w_1', \ldots, w_l')$ has positive determinant. By a similar argument orientation of $(b_1, \ldots, b_{k+l})$ does not depend on the choice of positively oriented bases of A and C. Therefore, orientation of B is determined by orientations of A and C. Let orientations of B and A be given. It is clear from Formula 12 that orientation of $(b_1, \ldots, b_{k+l})$ changes when orientation of $(w_1, \ldots, w_l)$ changes which is if and only if orientation of $(c_1, \ldots, c_l)$ changes. Therefore, orientation of C is determined. Similarly if orientations of B and C are given. $\square$

Remark 20 (Relation to the Direct Sum Orientation). Orientation of A, B, C consistently with a SES is the same as orientation of A, B, C so that the isomorphism $\psi : A \oplus C \to B$ prescribed by

$$(a_1, \ldots, a_k, c_1, \ldots, c_l) \mapsto (i(a_1), \ldots, i(a_k), w_1, \ldots, w_l)$$

is orientation preserving when $A \oplus C$ is equipped with the direct sum orientation. I chose the formalism of sequences because it keeps track of the maps involved. Note that the SES splits and the splitting maps give raise to an opposite SES $0 \to C \to B \to A \to 0$. The consistent orientation then corresponds to the requirement that $\psi : C \oplus A \to B$ given by $(c_1, \ldots, c_l, a_1, \ldots, a_k,) \mapsto (w_1, \ldots, w_l, i(a_1), \ldots, i(a_k))$ is an orientation preserving isomorphism. These two orientations are related by $A \oplus C \simeq (-1)^{\dim(A)\dim(C)} C \oplus A$.

Lemma 26 (Diagrams of Oriented Vector Spaces). *The following statements hold for oriented finite dimensional vector spaces:*

(a) Assume there is a commutative diagram of oriented vector spaces

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd} \tag{13}$$

where the rows are exact, the vector spaces are oriented consistently, and two of the vertical maps are both either orientation preserving or orientation reversing isomorphisms. Then the third map is an orientation preserving isomorphism.

(b) Assume there is a commutative diagram of oriented vector spaces

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & G \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd} \tag{14}$$

where all the columns and the two top / bottom rows are exact and all the spaces are oriented consistently. Then the bottom / top row is exact as well and the orientation induced on G / A from this row is $(-1)^{\dim(G)\dim(C)}$ times the orientation induced from the column. In other words, the diagram can be consistently oriented with respect to all rows and columns if and only if $(-1)^{\dim(G)\dim(C)} = 1$.

Proof. Conclusions about exactness are known as Five Lemma (a) and Nine Lemma (b) and we will not prove them. In fact, we will not even need it in the text. With orientations it works as follows:

As for (a), it is easy to see from commutativity that in the setting of Definition 26 the oriented basis $(i(a_1), \ldots, i(a_k), w_1, \ldots, w_l)$ of B is mapped to an oriented basis $(i(a'_1), \ldots, i(a'_k), w'_1, \ldots, w'_l)$ of B' under β where $a'_i = \alpha(a_i)$, $j'(w'_i) = c'_i$, $c'_i = \gamma(c_i)$. The claim is now obvious.

Consider the first case of part (b), the second case can be done similarly. By abuse of notation we will denote A, C, G, I some positively oriented bases of the corresponding spaces and $\iota_{XY} : X \to Y$ the maps between X, Y. Then $(\iota_{DE}\iota_{AD}(A), \iota_{DE}(G'), C', I')$ where $\iota_{DG}(G') = G$, $\iota_{EF}(C') = \iota_{CF}(C)$, $\iota_{FI}\iota_{EF}(I') = I$ is a positively oriented basis of the middle space E. Using commutativity we get $\iota_{EH}(\iota_{DE}(G'), I') = (\iota_{GH}(G), I'')$ where $\iota_{HI}(I'') = I$ and $(\iota_{DE}\iota_{DA}(A), C') = \iota_{BE}(\iota_{AB}(A), C'')$, $\iota_{BC}(C'') = C$. However, $(\iota_{AB}(A), C'')$ is a positively oriented basis of B and the basis $(\iota_{GH}(G), I'')$ of H has $(-1)^{\dim(G)\dim(C)}$ times the orientation of H as $(\iota_{DE}\iota_{DA}(A), \iota_{DE}(G'), C', I') = (-1)^{\dim(G)\dim(C)}(\iota_{DE}\iota_{DA}(A), C', \iota_{DE}(G'), I')$ and H is consistent with the middle row. We see that in order to orient H consistently with the bottom most sequence too, we have to take $(-1)^{\dim(G)\dim(C)}G$ instead of G. The lemma is hereby proven. $\square$

Remark 21 (Boundary Orientation by Inward Pointing Vector). Suppose $\dim(P) > 1$ and let $F \in \mathcal{F}^1(P)$ be equipped with the boundary orientation. Let $p \in P$ and take an inward pointing vector $v \in T_pP^+ \backslash T_pF$. We will work within a positively oriented chart at p which maps F into $\partial_i\mathbb{R}^n_+$. The oriented bases $(e_1, \ldots, e_n)$ and $(e_1, \ldots, e_{i-1}, v, e_{i+1}, \ldots, e_n)$ of T_pP are positively oriented since we can write $v = \lambda e_i + \sum_{j \neq i} \alpha_j e_j$ where $\lambda > 0$; thus, the determinant of the transition matrix is λ. We may calculate

$$\begin{aligned}
&(e_1, \ldots, e_{i-1}, v, e_{i+1}, \ldots, e_n) = (-1)^{i+1}(v, e_1, \ldots, \hat{e}_i, \ldots, e_n) = \\
&= v \times (-1)^{i+1}(e_1, \ldots, \hat{e}_i, \ldots, e_n)
\end{aligned}$$

where $(e_1, \ldots, \hat{e}_i, \ldots, e_n)$ is an oriented basis of T_pF whose orientation is $(-1)^{i+1}$ times the boundary orientation by remark 18. Therefore, the boundary orientation corresponds to orientation of T_pF through the direct sum

$$T_pP = \mathrm{span}\{v\} \oplus T_pF$$

where $\mathrm{span}\{v\}$ is positively oriented by v. If $\dim(P) = 1$ then we have $T_pP = o(p)\mathrm{span}\{v\} = \mathrm{span}\{v\} \times o(T_pF)$ by the definitions; hence, the same orientation rule applies.

LEMMA 27 (Orientation of Transverse Intersection). *Let P, Q, M be oriented manifolds with corners, $\partial M = \emptyset$, and let $f : P \to M$, $g : Q \to M$ be smooth transverse maps. For $(p,q) \in P \times_M Q$, $m = f(p) = g(q)$ there is a short exact sequence*

$$0 \to T_{(p,q)}(P \times_M Q) \xhookrightarrow{\mathrm{d}\iota_{(p,q)}} T_pP \oplus T_qQ \xrightarrow{\pi_m \circ \mathrm{d}(f\times g)_{(p,q)}} Z_m \to 0 \tag{15}$$

where $\iota : P \times_M Q \hookrightarrow P \times Q$ is the inclusion, $\Delta : M \to M \times M$ the diagonal embedding, and $\pi_m : T_mM \oplus T_mM \to Z_m$ the algebraic quotient map for

$$Z_m = T_mM \oplus T_mM/\mathrm{Im}(\mathrm{d}\Delta_m).$$

We assume both $T_mM \oplus T_mM$ and $T_pP \oplus T_qQ$ are equipped with the direct sum orientation. We orient Z_m consistently with the SES

$$0 \to T_mM \xhookrightarrow{\mathrm{d}\Delta_m} T_mM \oplus T_mM \xrightarrow{\pi_m} Z_m \to 0 \tag{16}$$

Then there is an orientation on $P \times_M Q$ such that $T_{(p,q)}(P \times_M Q)$ is oriented consistently with the SES 15 for every $(p,q) \in P \times_M Q$.

Proof. We check exactness of SES 15: The transversality criterion

$$\mathrm{d}f_p(C_pP) \oplus \mathrm{d}g_q(C_qQ) + \mathrm{d}\Delta_m(T_mM) = T_mM \oplus T_mM$$

implies that the map $C_pP \oplus C_qQ \to Z_m$ is surjective. In particular, the map $T_pP \oplus T_qQ \to Z_m$ is surjective. Therefore, the sequence is exact at Z_m. Exactness at $T_{(p,q)}(P \times_M Q)$ is obvious and it remains to show $T_{(p,q)}(P \times_M Q) = \mathrm{Ker}(\pi_m \circ \mathrm{d}(f \times g)_{(p,q)})$: The map $(f \times g) \circ \iota : P \times_M Q \to M \times M$ factors through $\Delta : M \to M \times M$ because $f = g$ on $P \times_M Q$. We see that $d(f \times g)_{(p,q)}(T_{(p,q)}(P \times_M Q)) \subset d\Delta_m(T_mM) = \mathrm{Ker}(\pi_m)$. Therefore, we get the inclusion " $\subset$ ". Dimension of Kernel and Image Formula gives $\dim(\mathrm{Ker}(\pi_m \circ d(f \times g)_{(p,q)})) = \dim(T_pP \oplus T_qQ) - \dim(Z_m) = \dim(P) + \dim(Q) - \dim(M) = \dim(T_{(p,q)}(P \times_M Q))$. Therefore, the sequence is exact at $T_pP \oplus T_qQ$ from dimensional reasons.

Referring back to the proof of Theorem 4 there is a positively oriented chart $\varphi : W \to \mathbb{R}^{\dim(M)}$ on M at m, and coordinate neighborhoods $U \subset P$, $V \subset Q$ of p, q, respectively, which restrict to a coordinate neighborhood $(P \times_M Q) \cap (U \times V)$ on $P \times_M Q$ such that when we define $v : W \times W \to \mathbb{R}^{\dim(M)}$ by $v(m,n) = \varphi(m) - \varphi(n)$ for all $m, n \in W$ and set $h = v \circ (f \times g) : U \times V \to \mathbb{R}^{\dim(M)}$ then h is a neat submersion on $U \times V$ and it holds

$$U \times_W V = h^{-1}(0).$$

Now for all $(p,q) \in U \times_W V$, $m = f(p) = g(q)$ we get exact commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & T_{(p,q)}(P \times_M Q) \arrow[r] \arrow[d, equal, "\mathrm{id}"] & T_pP \oplus T_qQ \arrow[r] \arrow[d, equal, "\mathrm{id}"] & Z_m \arrow[r] \arrow[d, "\gamma_m"] & 0 \\
0 \arrow[r] & T_{(p,q)}(U \times_W V) \arrow[r, "d\iota_{(p,q)}"] & T_pU \oplus T_qV \arrow[r, "dh_{(p,q)}"] & \mathbb{R}^{\dim(M)} \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & T_mM \arrow[r] \arrow[d, equal, "\mathrm{id}"] & T_mM \oplus T_mM \arrow[r] \arrow[d, equal, "\mathrm{id}"] & Z_m \arrow[r] \arrow[d, "\gamma_m"] & 0 \\
0 \arrow[r] & T_mW \arrow[r, "d\Delta_m"] & T_mW \oplus T_mW \arrow[r, "dv_{(m,m)}"] & \mathbb{R}^{\dim(M)} \arrow[r] & 0
\end{tikzcd}$$

where $\gamma_m : Z_m \to \mathbb{R}^{\dim(M)}$ is defined by $\gamma_m([(u,v)]) = d\varphi_m(u) - d\varphi_m(v)$ for all $[(u,v)] \in Z_m$. Now we equip spaces in the bottom sequences with orientations induced from charts and the spaces in the top sequences with orientations from the statement of the lemma. It follows from the fact that (W, φ) is positively oriented and from (a) of Lemma 25 applied to the second diagram that γ_m preserves orientation. When applied to the first diagram it follows that orientations of $T_{(p,q)}(U \times_W V)$ and $T_{(p,q)}(P \times_M Q)$ agree if and only if orientations of $T_pP \oplus T_qQ$ and $T_pU \oplus T_qV$ agree. Therefore, orientations either agree or disagree for all (p,q) in the chart domain. Points $(p,q) \in \mathrm{Int}(P \times_M Q)$ lie in $\mathrm{Int}(P) \times \mathrm{Int}(Q)$ because the neat embedding $P \times_M Q \subset P \times Q$ is depth-preserving by Theorem 4. As a result, we can always modify charts at $(p,q) \in \mathrm{Int}(P \times_M Q)$ on $P \times Q$ so that the orientations agree. Therefore, there is a positively oriented cover of $\mathrm{Int}(P \times_M Q)$ inducing the required orientation on tangent spaces. The lemma is hereby proven. $\square$

Definition 27 (Orientation on Transverse Intersection). *Let P, Q, M be oriented manifolds with corners, $\partial M = \emptyset$ and let $f : P \to M$, $g : Q \to M$ be smooth transverse maps. Then we define the* ***transverse orientation on*** *$P \times_M Q$ as ε-times the orientation consistent with the SES 15 where*

$$\varepsilon = (-1)^{\dim(M)(\dim(Q)+\dim(M))}$$

Convention 4. When orientation is considered, we will write $(-1)^P$ instead of $(-1)^{\dim(P)}$, for instance, by abuse of notation.

Remark 22 (Relation to an Other Method of Orientating $P \times_M Q$). The sign ε is a convention which is chosen such that theorems 13 and 14 below hold. In fact, it relates our method of orienting $P \times_M Q$ to a more general method of orienting fiber products explained in [1] which originally comes from [12]. In case when $f : P \to M$, $g : Q \to M$ are submersions this method orients $T_{(p,q)}(P \times_M Q)$ such that

$$T_{(p,q)}(P \times_M Q) \simeq \mathrm{Ker}(\mathrm{d}f_p) \oplus T_mM \oplus \mathrm{Ker}(\mathrm{d}g_q)$$

where

$$T_pP \simeq \mathrm{Ker}(\mathrm{d}f_p) \oplus \mathrm{Im}(\mathrm{d}f_p)$$
$$T_qQ \simeq \mathrm{Im}(\mathrm{d}g_q) \oplus \mathrm{Ker}(\mathrm{d}g_q)$$

and $\mathrm{Im}(\mathrm{d}\Delta_m) \simeq \mathrm{Im}(\mathrm{d}f_p) \simeq \mathrm{Im}(\mathrm{d}g_p) \simeq T_mM$ are all isomorphisms of oriented vector spaces. To see where ε arises we may calculate

$$\begin{aligned}
&T_{(p,q)}(P \times_M Q) \oplus Z_m \simeq \mathrm{Ker}(\mathrm{d}f_p) \oplus \mathrm{Im}(\mathrm{d}\Delta_m) \oplus \mathrm{Ker}(\mathrm{d}g_q) \oplus Z_m \simeq \\
&\simeq (-1)^{M(Q+M)}\mathrm{Ker}(\mathrm{d}f_p) \oplus \mathrm{Im}(\mathrm{d}\Delta_m) \oplus Z_m \oplus \mathrm{Ker}(\mathrm{d}g_q) \simeq \\
&\simeq \varepsilon\mathrm{Ker}(\mathrm{d}f_p) \oplus T_mM \oplus T_mM \oplus \mathrm{Ker}(\mathrm{d}g_q) \simeq \varepsilon T_pP \oplus T_qQ.
\end{aligned}$$

where all $\simeq$ are isomorphisms of oriented vector spaces.

Content of the following two theorems is the content of Lemma 3.1 and Corollary 3.2 in [1]. These results are all proven in [12] for manifolds with boundary using the different orientation convention above.

Theorem 13 (Boundary-Transverse Orientation). *Let P, Q, M be oriented manifolds with embedded faces, $\partial M = \emptyset$, and let $f : P \to M$, $g : Q \to M$ be smooth transverse maps. Then*

$$\tilde{\partial}(P \times_M Q) = (\tilde{\partial}P \times_M Q) \sqcup (-1)^{P+M}(P \times_M \tilde{\partial}Q) \tag{17}$$

Here $\tilde{\partial}$ is the abstract boundary defined in Definition 7 and the equality holds as an equality of oriented manifolds with embedded faces.

Proof. By Remark 11 the equality holds as an equality of non-oriented manifolds. To show that the orientation is correct let $H \in \mathcal{F}^1(P \times_M Q)$. There are two options:

Assume there is an $E \in \mathcal{F}^1(Q)$ such that $H \subset P \times_M E$. Since all orientations are well defined and the faces are connected it suffices to compare tangent space orientations at one particular point. We pick a $(p,q) \in \mathrm{Int}(H)$, $m = f(p) = g(q)$ and an inward pointing vector $v^+ \in T_{(p,q)}(P \times_M Q) \backslash T_{(p,q)} H$. Because $(p,q) \in \mathrm{Int}(H)$ we have $C_{(p,q)}(P \times_M Q) = T_{(p,q)}H$ and $(p,q) \in \mathrm{Int}(P) \times \mathrm{Int}(E)$ because the embedding $P \times_M Q \hookrightarrow P \times Q$ is depth-preserving; hence, we also have $C_{(p,q)}(P \times Q) = T_p P \oplus T_q E$. The embedding is also neat and condition (ii) of neatness asserts the induced map

$$T_{(p,q)}(P \times_M Q)/C_{(p,q)}(P \times_M Q) \to T_{(p,q)}(P \times Q)/C_{(p,q)}(P \times Q)$$

is injective. By Lemma 8 we have $\mathrm{d}\iota_{(p,q)}(T^+_{(p,q)}(P \times_M Q)) \subset T^+_{(p,q)}(P \times Q)$. Altogether, $\mathrm{d}\iota_{(p,q)}(v^+) \in T^+_{(p,q)}(P \times Q) \backslash T_{(p,q)}(P \times E) \simeq T_q^+ Q \backslash T_q E$. Hence, $\mathrm{d}\iota_{(p,q)}(v^+)$ can be considered as an inward pointing vector on Q too. We let $V = \mathrm{span}\{v^+\}$, $V' = \mathrm{span}\{\mathrm{d}\iota_{(p,q)}(v^+)\}$ and write down the following exact commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & V \arrow[r, equal, "\mathrm{d}\iota_{(p,q)}"] \arrow[d, hook] & V' \arrow[r] \arrow[d, hook] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{(p,q)}(P \times_M Q) \arrow[r, hook] \arrow[d] & T_p P \oplus T_q Q \arrow[r] \arrow[d] & Z_m \arrow[r] \arrow[d, equal, "\mathrm{id}"] & 0 \\
0 \arrow[r] & T_{(p,q)} H \arrow[r, hook] \arrow[d] & T_p P \oplus T_q E \arrow[r] \arrow[d] & Z_m \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The two bottom rows are the SESes of transverse orientation and the two columns from the left come from splitting $T_{(p,q)}(P \times_M Q) = V \oplus T_{(p,q)} H$, $T_p P \oplus T_q Q = V' \oplus (T_p P \oplus T_q E)$ and determine the boundary orientation according to Remark 21. Since there is 0 in the upper right corner (b) of Lemma 26 asserts the diagram can be oriented consistently. Now we suppose V, V' are oriented along the inward pointing vectors, and Z_m, $T_p P$, $T_q Q$ are oriented from the orientation of M, P, Q, respectively. The diagram then induces consistent orientation on the rest of spaces. However, the orientation induced on T_qE differs by $(-1)^P$ to the orientation of E as a boundary of Q since $T_pP \oplus T_qQ = V' \oplus T_pP \oplus T_qE = (-1)^P T_pP \oplus (V' \oplus T_qE)$. We also have to count in the convention for transverse orientation. Eventually, the total difference of orientation of $T_{(p,q)}H$ as $\partial(P \times_M Q)$ / as $P \times_M \partial Q$ and the consistent orientation from the diagram is

$$\delta = (-1)^{M(Q+1)} \qquad \text{and} \qquad \delta' = (-1)^{P+M(E+1)},$$

respectively. We see using $E = Q - 1$ that $\delta' = (-1)^{P+M}\delta$. However, this is what we were supposed to prove.

The second case, when $E \in \mathcal{F}^1(P)$, follows by the same reasoning from a similar diagram with the only difference that $T_pP \oplus T_qE$ is replaced by $T_pE \oplus T_qQ$. Orientation of $T_{(p,q)}H$ then gets the same sign $(-1)^{M(Q+1)}$ from the top and from the right. The theorem is hereby proven. □

Theorem 14 (Orientation and Operations on Transverse Intersections). *Let P, Q, R, P', Q', R', M, N be oriented connected manifolds with corners with $\partial M = \emptyset$, $\partial N = \emptyset$. Then the following claims hold:*

(a) (Associativity) Let $f : P \to M$, $h : R \to N$ and $g = (g^M, g^N) : Q \to M \times N$ be smooth maps such that $f \pitchfork g^M$ and $g^N \pitchfork h$. Then

$$(P \times_M Q) \times_N R = P \times_M (Q \times_N R)$$

as oriented manifolds with corners.

(b) (Commutativity) Let $f : P \to M$, $g : Q \to M$ be smooth maps such that $f \pitchfork g$ and let $\tau : P \times Q \to Q \times P$ be the canonical diffeomorphism

defined by $\tau(p,q) = (q,p)$ for all $(p,q) \in P \times Q$. Then under the induced diffeomorphism $\tau : P \times_M Q \to Q \times_M P$ it holds

$$P \times_M Q \simeq (-1)^{(P+M)(Q+M)} Q \times_M P$$

(c) *(Iteration) Let $f = (f^M, f^N) : P \to M \times N$, $g : P \to M$, $h : Q \to N$ be smooth maps such that $f^M \pitchfork g$ and $f^N \pitchfork h$. Then*

$$P \times_{M \times N} (Q \times R) = (-1)^{N(M+Q)} (P \times_M Q) \times_N R$$

as oriented manifolds with corners.

(d) *Let $f : P \to M$, $g : Q \to M$ be smooth maps such that $f \pitchfork g$ and let $\varphi_P : P \to P'$, $\varphi_Q : Q \to Q'$, $\varphi_M : M \to M'$ be diffeomorphisms. Let $\varepsilon(\varphi_X) = 1 \,/\, -1$ if the diffeomorphism with domain X is orientation preserving / reversing. If $\varphi_{P \times_M Q} : P \times_M Q \to P' \times_{M'} Q'$ is the induced diffeomorphism then*

$$\varepsilon(\varphi_{P \times_M Q}) = \varepsilon(\varphi_P)\varepsilon(\varphi_Q)\varepsilon(\varphi_M).$$

Proof. We will prove (a) first: Consider that all manifold we work with neatly embed in $P \times Q \times R$; thus, we will identify them with subspaces of $P \times Q \times R$. Then we have $(P \times_M Q) \times_N R = P \times_M (Q \times_N R)$ as subspaces and also as manifold with corners by Lemma 10. We denote this manifold by X. For every $(p,q,r) \in X$, $m = f(p) = g^M(m)$, $n = h(q) = g^N(n)$ we consider the commutative square

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_{(p,q,r)}X \arrow[r, hook] \arrow[d, hook] & T_pP \oplus T_{(q,r)}(Q \times_N R) \arrow[r] \arrow[d, hook] & Z_m^M \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & T_{(p,q)}(P \times_M Q) \oplus T_rR \arrow[r, hook] \arrow[d] & T_pP \oplus T_qQ \oplus T_rR \arrow[r] \arrow[d] & Z_m^M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_n^N \arrow[r, "id"] \arrow[d] & Z_n^N \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We assume all vector spaces except for $T_{(p,q,r)}X$ are oriented consistently with the diagram. Obviously, these orientations are not the same as transverse orientations because of the normalization factors. Moreover, the middle row does not correspond to the SES 15 which defines transverse orientation on $T_{(p,q)}(P \times_M Q)$ and an additional factor $(-1)^{MR}$ emerges. The overall difference of transverse orientation of X as $(P \times_M Q) \times_N R$ and the orientation consistent with the left most column of the diagram is

$$\delta = (-1)^{MR+N(R+1)+M(Q+1)}.$$

The same quantity for $P \times_M (Q \times_N R)$ and the top most row is

$$\delta' = (-1)^{M(Q+R-N+1)+N(R+1)}.$$

By (b) of Lemma 26, however, orientations of $T_{(p,q,r)}X$ from the left most column and the top most row differ by factor $(-1)^{MN}$. A simple calculation gives

$$\delta = (-1)^{MN}\delta'$$

and we deduce that both transverse orientations are equal. Therefore, (a) is proven.

We will prove (b): Note that τ induces a diffeomorphism of transverse intersections by (ii) of Remark 10. For every $(p,q) \in P \times_M Q$, $m = f(p) = g(q)$ we get the following exact commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T_{(p,q)}(P \times_M Q) \arrow[r] \arrow[d, "\mathrm{d}\tau_{(p,q)}"] & T_pP \oplus T_qQ \arrow[r] \arrow[d, "\mathrm{d}\tau_{(p,q)}"] & Z_m \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & T_{(q,p)}(Q \times_M P) \arrow[r] & T_qQ \oplus T_pP \arrow[r] & Z_m \arrow[r] & 0 \end{tikzcd}$$

where γ is induced from the exact commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T_mM \arrow[r, "\mathrm{d}\Delta_m"] \arrow[d, "\mathrm{id}"] & T_mM \oplus T_mM \arrow[r] \arrow[d, "(\mathrm{d}\tau')_{(m,m)}"] & Z_m \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & T_mM \arrow[r, "\mathrm{d}\Delta_m"] & T_mM \oplus T_mM \arrow[r] & Z_m \arrow[r] & 0 \end{tikzcd}$$

where $\tau' : M \times M \to M \times M$ is the diffeomorphism defined by $\tau'(m,n) = (n,m)$ for all $(m,n) \in M \times M$. In the last diagram, id is orientation preserving isomorphism; thus, by (a) of Lemma 26 γ preserves orientation if and only if $(\mathrm{d}\tau')_{(m,m)}$ does. Therefore, γ is orientation preserving if and only if $(-1)^M = 1$. Using this and the same argumentation we see that $\mathrm{d}\tau_{(p,q)} : T_{(p,q)}(P \times_M Q) \to T_{(q,p)}(Q \times_M P)$ is an orientation preserving isomorphism in the first diagram if and only if $(-1)^{M+PQ} = 1$. Putting this together with the transverse orientation convention we see

$$(-1)^{M+PQ+M(Q+1)+M(P+1)} = (-1)^{(P+M)(Q+M)}$$

which is what we wanted.

We will prove (c): We set $X = P \times_{M \times N} (Q \times R) = (P \times_M Q) \times_N R$. For every $p \in P$, $q \in Q$, $r \in R$, $m = f^M(p) = g(q)$, $n = f^N(p) = h(r)$ we get an exact commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_{(p,q,r)}X \arrow[r] \arrow[d, equal, "\mathrm{id}"] & T_{(p,q)}(P \times_M Q) \oplus T_r R \arrow[r] \arrow[d] & T_n N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{(p,q,r)}X \arrow[r] \arrow[d] & T_p P \oplus T_q Q \oplus T_r R \arrow[r] \arrow[d] & T_m M \oplus T_n N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & T_m M \arrow[r] \arrow[d] & T_m M \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

This time, all spaces can be oriented consistently because there is 0 in the left-bottom corner. Therefore, the two occurrences of $T_{(p,q,r)}X$ are oriented the same from the diagram. It remains to compare to standard orientations and transverse orientations. There is a factor $(-1)^{MN}$ coming from the right most column and $(-1)^{MR}$ coming from the middle column. Altogether with the convention we get

$$\delta = (-1)^{MN+(M+N)(Q+R+1)} \qquad \text{and} \qquad \delta' = (-1)^{MR+N(R+1)+M(Q+1)}$$

for $X = P \times_{M \times N} (Q \times R)$ and $X = (P \times_M Q) \times_N R$, respectively. By comparing we get

$$\delta = (-1)^{N(M+Q)}\delta'$$

which was to be proven.

Proof of (d) is a simple usage of (a) of Lemma 26. For every $(p,q) \in P \times Q$, $m = f(p) = g(q)$, $p' = \varphi_P(p)$, $q' = \varphi_Q(q)$, $m' = \varphi_M(m)$ we have

$$\begin{tikzcd}
0 \arrow[r] & T_{(p,q)}(P \times_M Q) \arrow[r] \arrow[d, "\mathrm{d}(\varphi_P \times \varphi_Q)_{(p,q)}"] & T_p P \oplus T_q Q \arrow[r] \arrow[d, "\mathrm{d}(\varphi_P \times \varphi_Q)_{(p,q)}"] & Z_m \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & T_{(p',q')}(P' \times_M Q') \arrow[r] & T_{p'} P' \oplus T_{q'} Q' \arrow[r] & Z'_{m'} \arrow[r] & 0
\end{tikzcd}$$

where the map γ is given by

$$\begin{tikzcd}
0 \arrow[r] & T_mM \arrow[r, "\mathrm{d}\Delta_m"] \arrow[d, "(\mathrm{d}\varphi_M)_m"] & T_mM \oplus T_mM \arrow[r] \arrow[d, "\mathrm{d}(\varphi_M \times \varphi_M)_{(m,m)}"] & Z_m \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & T_{m'}M' \arrow[r, "\mathrm{d}\Delta'_{m'}"] & T_{m'}M' \oplus T_{m'}M' \arrow[r] & Z'_{m'} \arrow[r] & 0
\end{tikzcd}$$

In the last diagram, the vertical map in the middle is an orientation preserving isomorphism. Therefore, $\varepsilon(\gamma) = \varepsilon(\varphi_M)$. The first diagram gives $\varepsilon(\varphi_{P\times_M Q}) = \varepsilon(\varphi_P)\varepsilon(\varphi_Q)\varepsilon(\gamma)$ and the claim is proven. $\square$

2.10 Collar Neighborhoods

In this section we will show that 1-faces of a compact manifold with embedded faces have collar neighborhoods which are compatible near the common k-faces. In order to do this we will revise the theory of vector fields and flows for boundary-less manifolds from [4, p. 434-473] and generalize it so that integral curves are allowed to start and end at the boundary or to stay in the boundary at all times.

Definition 28 (Integral Curve). *Let P be a manifold with corners, $X : P \to TP$ a smooth vector field and $(p_0, t_0) \in P \times \mathbb{R}$. An* ***integral curve of X at*** *(p_0, t_0) is a smooth curve $\gamma : I \to P$ where I is a non-trivial interval containing t_0 such that $\gamma(t_0) = p$ and for every $t \in I$ it holds*

$$\dot{\gamma}(t) = X(\gamma(t))$$

It is called ***maximal*** *if: Whenever $\tilde{\gamma} : \tilde{I} \to P$ is an integral curve and there exists a $t \in I \cap \tilde{I}$ such that $\gamma(t) = \tilde{\gamma}(t)$ then $\tilde{I} \subset I$ and $\gamma = \tilde{\gamma}\big|_{\tilde{I}}$.*

Recall that integral curves on a boundary-less manifold are defined on open intervals only. Clearly, if P is embedded in Q and the vector field X on P is a restriction of a vector field $\tilde{X}$ on Q then integral curves of X are precisely integral curves of $\tilde{X}$ which lie in P. Therefore, some results might follow directly from the theory for manifolds without boundary since manifolds with corners embed, at least locally, into manifolds without boundary. In this text, however, a separated ODE theorem for Euclidean corners 23 is formulated in the Appendix and used as a local result.

Lemma 28 (Uniqueness of Integral Curves). *Let P be a manifold with corners, $X : P \to TP$ a smooth vector field and let $\gamma : I \to P$, $\tilde{\gamma} : \tilde{I} \to P$ be two integral curves of X. If there is an $t_0 \in I \cap \tilde{I}$ such that $\gamma(t_0) = \tilde{\gamma}(t_0) = p_0$ then $\gamma = \tilde{\gamma}$ on $I \cap \tilde{I}$.*

Proof. Denote $J = I \cap \tilde{I}$. If $J = \{t_0\}$ then the conclusion holds. Suppose there is a $t_1 \in J$ such that $t_1 > t_0$. We will 1 prove that γ and $\tilde{\gamma}$ are equal on $[t_0, t_1]$. The same can be done if $t_1 < t_0$ and the lemma follows as every $t \in J$ is contained in some of these intervals. We may restrict γ, $\tilde{\gamma}$ and assume $J = I = \tilde{I} = [t_0, t_1]$. Let $t \in J$, denote $\gamma(t) = \tilde{\gamma}(t) = p$, and let (U, φ) be a chart at p. Because the curves are continuous, there is a non-trivial interval $K \subset J$, $t \in K$ which is open as a subspace of J and is mapped into U by γ, $\tilde{\gamma}$. Both $\gamma\big|_K$, $\tilde{\gamma}\big|_K : K \to U$ then solve the initial value problem for X at (p, t) in coordinates. The ODE Theorem 23 asserts the curves are equal on K. We see $\{\gamma = \tilde{\gamma}\} \subset J$ is open, closed and non-empty as $\gamma(t_0) = \tilde{\gamma}(t_0)$; thus, it equals J. $\square$

Remark 23 (Unique Maximal Integral Curve). Let $(p_0, t_0) \in P \times \mathbb{R}$. We define I as the union of domains of all integral curves of X at (p_0, t_0). If $t \in I$ then there is an integral curve $\tilde{\gamma} : \tilde{I} \to P$ with $t \in \tilde{I}$ and we define $\gamma(t) = \tilde{\gamma}(t)$. This is a well defined smooth curve $\gamma : I \to P$ by the previous lemma. If $I \neq \emptyset$ then this curve is the unique maximal curve at (p_0, t_0) by definition and, in fact, the unique maximal integral curve at all its points .

Existence of integral curves starting at boundary points is not guaranteed for a general smooth vector field. This occurs already for manifold with boundary as it is shown in the example below. A notion of compatibility of a vector field with the boundary has to be introduced.

Example 7 (Vector Fields not Compatible with Boundary). Consider the smooth function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} \sin\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Define a smooth vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$ by the formula

$$X(x, y) = (1, f'(x))$$

We look for an integral curve $\gamma : I \to \mathbb{H}^2$ of the vector field $X\big|_{\mathbb{H}^2}$ which satisfies $\gamma(0) = (0, 0)$. By integration we get $\gamma(t) = (t, f(t))$ for every $t \in I$. However, in every neighborhood of 0 in I a negative value is attained by $f(t)$. Therefore, there is no integral curve of X on $\mathbb{H}^2$ starting at $(0, 0)$.

When corners of higher codimension are allowed there is a trivial counterexample; namely, the constant vector field $X = (-1, 1)$ on $\mathbb{R}^2_+$ does not have any integral curve at $(0, 0)$.

Figure 4: These are the situations from Example 7. Points at which there is no integral curve are highlighted.

Definition 29 (Vector Field Compatible with Boundary). *Let P be a manifold with embedded faces and $X : P \to TP$ a smooth vector field. Then X is said to be* ***compatible with boundary*** *if for every $F \in \mathcal{F}^1(P)$ one of the following situations occurs:*

(a) X is tangent to F; i.e. $X_p(p) \in T_pF$ for all $p \in F$

(b) X is inward pointing on F; i.e. $X_p(p) \in T_p^+P \backslash T_pF$ for all $p \in F$

(c) X is outward pointing on F; i.e. $X_p(p) \in -T_p^+P \backslash T_pF$ for all $p \in F$

If $p \in P$ and there exists a face $F \in \tilde{F}^1_p(P)$ such that X is inward pointing / outward pointing on F then we say X is ***inward pointing / outward pointing at*** *p.*

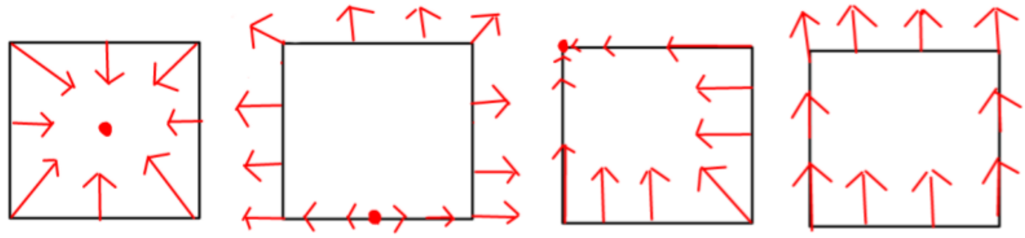

Figure 5: These are examples of vector fields compatible with boundary on the square. The dot is a point where the vector field vanishes. Note that the compatibility condition guarantees it can not happen that the vector field is inward pointing and outward pointing on two adjacent faces.

Lemma 29 (Existence of Integral Curves Compatible With Boundary). *Let P be a manifold with embedded faces and $X : P \to TP$ a smooth vector field compatible with boundary. Then for every $(p, t) \in P \times \mathbb{R}$ there is an integral curve at (p, t). Moreover, if X is tangent to every $F \in \tilde{\mathcal{F}}^1_p(P)$ then an integral curve exists on an open interval containing t.*

Proof. Direct consequence of ODE Theorem 23 applied in coordinates. □

Lemma 30 (Form of Integral Curves Compatible With Boundary). *Let P be a manifold with embedded faces and $F \in \mathcal{F}^1(P)$. Let $X : P \to TP$ be a smooth vector field compatible with boundary and let $\gamma : I \to P$ be an integral curve of X. Denote $I_F = \{t \in I : \gamma(t) \in F\}$. Then one of the following mutually exclusive cases holds:*

(a) $I_F = \emptyset$

(b) X is tangent to F and $I_F = I$

(c) X is inward pointing on F and $I_F = \{t_0\}$ is a left end-point of I

(d) X is outward pointing on F and $I_F = \{t_1\}$ is a right end-point of I

Proof. The ODE Theorem 23 gives local statements (c), (d) and openness of $I_F \subset I$ in (b). I_F is also closed and non-empty; thus, $I_F = I$. □

A subtle change applies also to the definition of a flow; namely, its domain is no longer an open subspace of $M \times \mathbb{R}$ as it is when $\partial M = \emptyset$. Therefore, the flow should be considered rather as a family of integral curves. However, it is still smooth in the sense of definition 4 and the most important properties hold.

DEFINITION 30 (Flow of a Vector Field). *Let P be a manifold with corners and $X : P \to TP$ a smooth vector field. A **flow of** X **on** P is a smooth map $\psi : W \to P$ with domain $W \subset P \times \mathbb{R}$ such that for every $p \in P$ the subspace $W_p = \{t \in \mathbb{R} : (p,t) \in W\}$ is either empty or a non-trivial interval containing 0. In the latter case, the map*

$$\psi_p : t \in W_p \mapsto \psi(p,t) \in P$$

*is an integral curve of X at $(p,0)$. For a $t \in \mathbb{R}$ we also denote $W^t = \{p \in P : (p,t) \in W\}$ and define $\psi^t : W^t \to P$ by the formula $\psi^t(p) = \psi(p,t)$ for all $p \in W^t$. We call this the **fixed-time flow** of X.*

*A flow ψ of X is called **maximal** if: Whenever $\psi' : W' \to P$ is a flow of X then $W' \subset W$ and $\psi' = \psi\big|_{W'}$.*

THEOREM 15 (Maximal Flow Existence Theorem). *Let P be a manifold with embedded faces and $X : P \to TP$ a smooth vector field compatible with boundary. Then there exists a unique maximal flow $\psi : W \to P$. In addition, for every $p \in P$ holds*

(a) $\psi_p : W_p \to P$ is a maximal integral curve at $(p,0)$.

(b) *For every $s \in W_p$ we have $W_{\psi(p,s)} = W_p - s$ and for every $t \in W_{\psi(p,s)}$ it holds*

$$\psi(\psi(p,s),t) = \psi(p,s+t)$$

(c) *There exists an open neighborhood $U \subset P$ of p and an $\varepsilon > 0$ such that $U \times (-\varepsilon, \varepsilon)$ / $U \times [0, \varepsilon)$ / $U \times (-\varepsilon, 0] \subset W$ according to whether X is tangent to all faces / inward pointing / outward pointing at p, respectively.*

Proof. We define W to be the union $\bigcup_{p \in P} \{p\} \times I_p$ where I_p is domain of the maximal curve γ at p as in Remark 23. If $t \in I_p$ we set $\psi(p,t) = \gamma(t)$. By Lemma 29 there is an integral curve at every point; hence, $I_p \neq \emptyset$ for every $p \in P$. Therefore, (a) holds.

In order to prove (b), fix $s \in W_p$ and define $\gamma(t) = \psi(p, s+t)$ for all $t \in W_p - s$. Then γ is an integral curve at $(\psi(p,s), 0)$. By the definition of W and ψ we have $W_p - s \subset W_{\psi(p,s)}$ and $\gamma = \psi_{\psi(p,s)}$ on the common domain. On the other hand, we may define $\tilde{\gamma}(t) = \psi_{\psi(p,s)}(t-s)$ for all $t \in W_{\psi(p,s)} + s$. This is an integral curve at $(p, 0)$ as $\tilde{\gamma}(0) = \psi(\psi(p,s), -s) = \gamma(-s) = p$. We have $W_{\psi(p,s)} + s \subset W_p$ by the definition of W. Therefore, $W_p - s = W_{\psi(p,s)}$ and the claim follows.

As for (a), let $\tilde{\gamma} : \tilde{I} \to P$ be an integral curve and $s \in \tilde{I} \cap W_p$ such that $\tilde{\gamma}(s) = \psi_p(s)$ for some $p \in P$. If we define $\tilde{\gamma}'(t) = \tilde{\gamma}(s+t)$ for $t \in \tilde{I} - s$ we get an integral curve with $\tilde{\gamma}'(0) = \psi_{\psi(p,s)}(0)$. It follows from the definition of W and ψ that $\tilde{I} - s \subset W_{\psi(p,s)}$ and $\tilde{\gamma}' = \psi_{\psi(p,s)}$. However, $W_{\psi(p,s)} = W_p - s$ and $\psi_{\psi(p,s)}(t) = \psi(p, t+s)$ by (b); thus, $\tilde{I} \subset W_p$ and $\tilde{\gamma} = \psi_p$. Maximality of the flow is a direct consequence.

In this paragraph we will prove that $\psi : W \to P$ is smooth. Firstly, we will show it is smooth on a neighborhood of $P \times \{0\}$ in W. Let $p_0 \in P$. There is a regular chart (U, φ) at p_0, a manifold without boundary $\tilde{U}$ such that

$U \subset \tilde{U}$ is an embedded submanifold and a smooth vector field $\tilde{X} : \tilde{U} \to T\tilde{U}$ which restricts to X on U. This is just a reformulation of the situation in coordinates. By the Fundamental Theorem on Flows [4, p. 442, Theorem 17.8] there is a smooth maximal flow $\tilde{\psi} : \tilde{W} \to \tilde{U}$ of $\tilde{X}$ where $\tilde{W} \subset \tilde{U} \times \mathbb{R}$ is an open subspace. If we define $W' = \{(p,t) \in U \times \mathbb{R} : \exists$ integral curve $\gamma : I \to U$ of $X, \gamma(0) = p, t \in I\} \subset W$ then $W' \subset \tilde{W}$ and $\psi = \tilde{\psi}\big|_{W'}$ because every integral curve of X can be extended to an integral curve of $\tilde{X}$ as it is argued in the proof of Theorem 23. Therefore, we have a subspace $W' \subset W$ containing $(p_0, 0)$ such that $\psi' = \psi|_{W'}$ is smooth. We may also rewrite W' as $W' = \{(p,t) \in W : p \in U, \psi(p, I_t) \in U\}$ where $I_t = [0,t]$ / $[t,0]$ depending on the sign of t. Now it suffices to show that W' is a neighborhood of $(p_0, 0)$ in W:

For the contrary, assume there is a sequence $(p_n, t_n) \in W \backslash W'$, $n \in \mathbb{N}$ such that $(p_n, t_n) \mapsto (p_0, 0)$. We set $\delta(p) = \sup -W'_p$, $\varepsilon(p) = \sup W'_p$ for $p \in U$. Since $p_n \mapsto p_0$ we may assume $p_n \in U$ for all $n \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ either $t_n \leq -\delta(p_n)$ or $\varepsilon(p_n) \leq t_n$ according to the sign of t_n and there is infinitely many of $n \in \mathbb{N}$ such that one of the two cases holds. Therefore, we may assume $t_n \geq \varepsilon(p_n)$ for all $n \in \mathbb{N}$ and proceed similarly in the second case. Since $t_n \in W_{p_n}$ and W_{p_n} is an interval it follows $\varepsilon(p_n) \in W_{p_n}$. It also holds that $\delta(p_n) \mapsto 0$. Since ψ' is continuous we may find an open neighborhood $V \subset U$ of p_0 and an $\eta > 0$ such that $V \times (-\eta, \eta) \cap W'$ is mapped into U' by ψ'. Here we recall that U' is a ball within the regular neighborhood U such that $\overline{U}' \subset U$. Assume $n \in \mathbb{N}$ sufficiently large such that $(p_n, \varepsilon(p_n)) \in V \times (-\eta, \eta)$. For all $0 \leq t < \varepsilon(p_n)$ we have $\psi(p_n, I_t) \subset U'$. Therefore, by continuity of the integral curve $\psi_{p_n} : W_{p_n} \to P$ we get $\psi(p_n, \varepsilon(p_n)) \in \overline{U}' \subset U$. In particular, $(p_n, \varepsilon(p_n)) \in W'$. There is a neighborhood J of $(p_n, \varepsilon(p_n))$ in W_{p_n} which is mapped into U. Because $(p_n, t_n) \notin W'$ we must have $\varepsilon(p_n) < t_n$ and we see that J has to contain an interval of the form $[\varepsilon(p_n), \varepsilon(p_n) + \delta]$ with $\delta > 0$. We have $\psi(p_n, [\varepsilon(p_n), \varepsilon(p_n) + \delta]) \subset U$; thus, $(p_n, \varepsilon(p_n) + \delta) \in W'$ by the definition of W'. However, this is a contradiction with the choice of $\varepsilon(p_n)$. We conclude, W' is a neighborhood of $(p_0, 0)$ in W. Therefore, ψ is indeed smooth on a neighborhood of $P \times \{0\}$ in W.

Secondly, we will prove that smoothness of ψ on a neighborhood of $P \times \{0\}$ in W and the group law (c) imply smoothness of ψ on the whole of W. Let $S \subset W$ be the subspace of all $(p, t) \in W$ such that there exists a neighborhood $W' \subset W$ of (p, t) such that $\psi\big|_{W'}$ is smooth. This is an open subspace and $\psi\big|_S$ is smooth. We will prove S equals the whole W. Let $p_0 \in P$ and let $t_0 \in W_{p_0}$ with $t_0 > 0$ such that $(p_0, t_0) \notin S$. The case $t_0 < 0$ is analogous. Set $s = \sup\{t \in W_{p_0} : (p_0, t) \in S\}$. We have $0 \leq s \leq t_0$; hence, $s \in W_{p_0}$. By the assumption S is a neighborhood of $(p_0, 0)$ in W. Because $[0, t_0] \subset W_{p_0}$ a subspace of the form $\{p_0\} \times [0, \delta)$ for some $\delta > 0$ has to be contained in S; thus, $s > 0$. Because S is a neighborhood of $(\psi(p_0, s), 0)$ in W we get an open neighborhood $U \subset P$ of $\psi(p_0, s)$ and $\varepsilon > 0$ such that $U \times (-\epsilon, \epsilon) \cap W \subset S$. Continuity of $\psi\big|_S$ guarantees $\psi\big|_S^{-1}(U) \subset S \subset W$ is open. Because $\psi(p_0, s) \in U$, $\psi_{p_0} : W_{p_0} \to P$ is continuous and $(p_0, t) \in S$ for every $0 \leq t < s$ we may find an $r > 0$ such that $s - \varepsilon < r < s$ and $(p_0, r) \in \psi\big|_S^{-1}(U)$. Since $\psi\big|_S^{-1}(U) \subset W$ is open there is an open neighborhood $V \subset P$ of p_0 such that $V \times \{r\} \cap W \subset \psi\big|_S^{-1}(U)$. We may write $\psi(p, t) = \psi(\psi(p, r), t - r)$ for all $(p, t) \in W' = V \times (r - \varepsilon, r + \varepsilon) \cap W$ because $W_{\psi(p,r)} = W_p - r$ and the group law holds by (c). Because $V \times \{r\} \cap W \subset \psi\big|_S^{-1}(U)$ the inner map $(p, t) \in W' \mapsto (\psi(p, r), t - r) \in P \times \mathbb{R}$ is smooth and its image is contained in $U \times (-\varepsilon, \varepsilon) \cap W$. However, $U \times (-\varepsilon, \varepsilon) \cap W \subset S$ and we see the composition of the inner map with ψ is smooth as well. Since W' is a neighborhood of (p_0, s) in W and $\psi\big|_{W'}$ is smooth we get $(p_0, s) \in S$. We must have $t_0 > s$

as $(p_0, t_0) \notin S$. Because $[s, t_0] \subset W_{p_0}$ and $S \subset W$ is open there is a $\delta > 0$ such that $\{p_0\} \times [s, s+\delta) \subset S$ which is a contradiction with the choice of s. Altogether $S = W$ and the flow $\psi : W \to P$ is smooth.

Condition (c) follows from Theorem 23 applied in coordinates to the neighborhood W' as in the proof of smoothness on a neighborhood of $P \times \{0\}$. □

THEOREM 16 (Complete Flows). *Let P be a manifold with embedded faces and $X : P \to TP$ a vector field with compact support which is compatible with boundary. Let $\psi : W \to P$ be its maximal flow. Then it holds: X is tangent to all faces / not outward pointing on any face / not inward pointing on any face if and only if $W = P \times \mathbb{R}$ / $P \times [0, \infty) \subset W$ / $P \times (-\infty, 0] \subset W$. We say ψ is complete / "+"-complete / "−"-complete, respectively.*

Proof. We will prove just the second case. The third case is similar and the first case is a combination of both. The "if" part follows from Lemma 30. The "only if" part is as follows :

If X is not outward pointing and $p \in P$ we use (c) of Theorem 15 to get an open neighborhood $U_p \subset P$ of p and $\varepsilon_p > 0$ such that $U_p \times [0, \varepsilon_p) \subset W$. Since $\operatorname{supp}(X)$ is compact we get an $\varepsilon > 0$ such that $\operatorname{supp}(X) \times [0, \varepsilon) \subset W$. The integral curves at points outside of $\operatorname{supp}(X)$ are constant; hence, $P \times [0, \varepsilon) \subset W$. To get a contradiction assume $s = \sup W_p < \infty$ for some $p \in P$. We may concatenate ψ_p with an integral curve $\psi_{\psi(p,t)} : [0, \varepsilon) \to P$ where $s - \varepsilon < t < s$ to get a new integral curve defined beyond s. This is a contradiction with the choice of s. □

Having the flow we may define the notion of Lie derivative similarly as in the boundary-less case.

Definition 31 (Lie Derivative). *Let P be a manifold with embedded faces and let $X, Y : P \to TP$ be smooth vector fields, X compatible with boundary. Let $\psi : W \to P$ be the maximal flow of X. Then for every $p \in P$ we define the **Lie derivative of** Y **along** X **at** p by the formula*

$$(\mathcal{L}_X Y)_p = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\mathrm{d}\psi^t)_p^{-1} Y_{\psi(p,t)}$$

Remark 24 (Fixed-time Flows). (i) Consider the fixed-time flows $\psi^t : W^t \to P$ for $t \in \mathbb{R}$. By the group law, (b) of Theorem 15, we have $\psi^t(W^t) = W^{-t}$ and also that $\psi^t : W^t \to W^{-t}$ is a bijection with an inverse $\psi^{-t} : W^{-t} \to W^t$. Both maps are smooth in the sense of Definition 4. Therefore, (iii) of Remark 10 asserts: If $W^t \subset P$ is an embedded submanifold with corners then $W^{-t} \subset P$ is an embedded submanifold with corners as well and $\psi^t : W^t \to W^{-t}$ is a diffeomorphism.

(ii) Let $p \in P$. By (c) of Theorem 15 there is an open neighborhood $U \subset P$ of p and a non-trivial interval I containing 0 such that $U \times I \subset W$. Therefore, $U \subset W^t$ and (i) implies that $V = \psi^t(U) \subset W^{-t}$ is an embedded submanifold of P. Because of this, we may differentiate the relation $\psi^{-t} \circ \psi^t = \mathrm{id}_{W^t}$ at p using the chain rule on maps $U \xrightarrow{\psi^t} V \xrightarrow{\psi^{-t}} U$. We get that $(\mathrm{d}\psi^t)_p$ is invertible with inverse $(\mathrm{d}\psi^{-t})_{\psi(p,t)}$. It is also smooth in t as a matrix in coordinates because ψ is smooth; hence, the Lie derivative is well defined.

Lemma 31 (Commuting Vector Fields and Flows). *Let P be a manifold with embedded faces and let X, $Y : P \to TP$ be smooth vector fields compatible with boundary. Let $\psi : W \to P$, $\varphi : W' \to P$ be their maximal flows. Then the following conditions are equivalent*

(a) $(\mathcal{L}_X Y)_p = 0$ *for all* $p \in P$.

(b) $(\mathcal{L}_Y X)_p = 0$ *for all* $p \in P$.

(c) For every $p \in P$ there is an open neighborhood $U \subset P$ of p and non-trivial intervals I, J containing 0 such that

$$\psi(\varphi(q,s),t) = \varphi(\psi(q,t),s)$$

for every $q \in U$ and $(t,s) \in I \times J$.

Proof. We will prove the implication (a) to (c) first : Assume there is an open neighborhood $\tilde{V} \subset P$ of p and $\delta > 0$ such that $\tilde{V} \times [0,\delta) \subset W$. If $\tilde{V} \times (-\delta, 0] \subset W$ then proceed analogously. In addition, take an open neighborhood $V \subset \tilde{V}$ of p and a non-trivial interval I containing 0 such that $V \times I \subset W$, $\mathrm{d}\psi_q^t$ exists and is invertible for $(q,t) \in V \times I$, and $\psi(V \times I) \subset \tilde{V}$. We may also find an open neighborhood $U \subset V$ of p and a non-trivial interval J containing 0 such that $U \times J \subset W'$ and $\varphi(U \times J) \subset V$. Fix $q \in U$, $t \in I$ and define a smooth curve by the formula $\gamma(s) = \psi(\varphi(q,s),t)$ for all $s \in J$. We have $\frac{\mathrm{d}}{\mathrm{d}s}\gamma(s) = (\mathrm{d}\psi^t)_{\varphi(q,s)} Y_{\varphi(q,s)}$ by the chain rule. For $r \in V$ and $u \in I$ we calculate using the group law

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}u}(\mathrm{d}\psi^u)_r^{-1} Y_{\psi(r,u)} = \frac{\mathrm{d}}{\mathrm{d}z}\bigg|_{z=0} (\mathrm{d}\psi^{u+z})_r^{-1} Y_{\psi(r,u+z)} = \frac{\mathrm{d}}{\mathrm{d}z}\bigg|_{z=0} (\mathrm{d}\psi^z \mathrm{d}\psi^u)_r^{-1} Y_{\psi(r,u+z)} = \\
&= (\mathrm{d}\psi^u)_r^{-1} \frac{\mathrm{d}}{\mathrm{d}z}\bigg|_{z=0} (\mathrm{d}\psi^z)^{-1} Y_{\psi(r,u+z)} = (\mathrm{d}\psi^u)_r^{-1} (\mathcal{L}_X Y)_{\psi(r,u)} = 0
\end{aligned} \tag{18}$$

where we apply the chain rule on $V \xrightarrow{\psi^u} \tilde{V} \xrightarrow{\psi^z} P$ for $0 \leq z < \delta$ on the first line. This equation is valid for all $u \in I$; hence, $(\mathrm{d}\psi^u)_r^{-1} Y_{\psi(r,u)} = (\mathrm{d}\psi^0)_r^{-1} Y_{\psi(r,0)} = Y_r$. In other words, $Y_{\psi(r,u)} = (\mathrm{d}\psi^u)_r Y_r$. In particular, when we set $r = \varphi(q,s)$, $u = t$ we have $\frac{\mathrm{d}}{\mathrm{d}s}\gamma(s) = Y_{\psi(\varphi(q,s),t)} = Y_{\gamma(s)}$. Therefore, γ is an integral curve of Y with $\gamma(0) = \psi(q,t)$; thus, $\gamma(s) = \varphi(\psi(q,t),s)$ for all $s \in J$ as $\varphi_{\psi(q,t)}$ is the maximal integral curve at $\psi(q,t)$. The implication (b) to (c) is analogous, exchanging X and Y.

The implication (c) to (a), and similarly the implication (c) to (b), is simple : Firstly, we take derivative of $\psi(\varphi(p,s),t) = \varphi(\psi(p,t),s)$ along s at

$s = 0$ to get $(\mathrm{d}\psi^t)_p Y_p = Y_{\psi(p,t)}$. For t small we write $Y_p = (\mathrm{d}\psi^t)_p^{-1} Y_{\psi(p,t)}$ and derive along t at $t = 0$ to get $(\mathcal{L}_X Y)_p = 0$. $\square$

Definition 32 (Commuting Vector Fields). *Let P be a manifold with embedded faces and X, $Y : P \to TP$ smooth vector fields compatible with boundary. If $(\mathcal{L}_X Y)_p = 0 = (\mathcal{L}_Y X)_p$ for all $p \in P$ we say X* ***and*** *Y* ***commute****.*

Remark 25 (Vector Field Commutes with Itself). A compatible vector field X commutes with itself since the group law holds and it is easy to check that condition (c) of the previous lemma applies.

In order to handle systems of vector fields we define the following notion:

Definition 33 (System of Neighborhoods). *Let P be a manifold with embedded faces and $\mathcal{G} \subset \mathcal{F}^1(P)$. Then a* ***system of neighborhoods of*** *$\mathcal{G}$ is an indexed set $\{U_G : G \in \mathcal{G}\}$ where $U_G \subset P$ is an open neighborhood of G for every $G \in \mathcal{G}$. If $F \in \mathcal{F}^1(P)$ we denote*

$$\mathcal{G}_F = \bigcup_{\substack{G \in \mathcal{G} \\ G \neq F}} \mathcal{F}^1(F) \cap \mathcal{F}^1(G)$$

and call it ***the boundary of*** *F* ***in*** *$\mathcal{G}$. For every $H \in \mathcal{G}_F$ we denote G_H the unique $G_H \in \mathcal{G}_H$ such that $H \in \mathcal{F}^1(G_H) \cap \mathcal{F}^1(F)$.*

Definition 34 (Normal Vector Fields). *Let P be a manifold with embedded faces and $F \in \mathcal{F}^1(P)$. A smooth vector field $X : U \to TU$ is called* ***normal to*** *F if $U \subset P$ is an open neighborhood of F in P, and X is inward pointing on F and tangent to all other 1-faces of P.*

A ***system of commuting normal vector fields*** *is an indexed set $\{X_G : G \in \mathcal{G}\}$ for some $\mathcal{G} \subset \mathcal{F}^1(P)$ such that X_G is normal to G for every $G \in \mathcal{G}$ and commutes with $X_{G'}$ on $U_G \cap U_{G'}$ for every $G' \in \mathcal{G}$.*

Remark 26 (Restrictions of Commuting Normal Vector Fields). In the situation of the definition above let $F \in \mathcal{F}^1(P)$. For every $E \in \mathcal{G}_F$ we define $U_E = U_{G_E} \cap F$ and $X_E = X_{G_E}\big|_F$. Then $X_E : U_E \to TG_E$ is a well defined smooth vector field as X_{G_E} is tangent to F and F is embedded in P. It is easy to see from Lemma 30 that for the flow $\psi_E : W_E \to F$ of X_E holds $W_E = W_{G_E} \cap F \times \mathbb{R}$, $\psi_E = \psi_{G_E}\big|_{W_E}$ where ψ_{G_E} is the flow of X_{G_E}. It follows that $(\mathcal{L}_{X_E} X_{E'})_p = (\mathcal{L}_{X_{G_E}} X_{G_{E'}})_p$ in T_pP if $p \in F$ and $E, E' \in \mathcal{G}_F$; hence, $\{X_E : E \in \mathcal{G}_F\}$ is a system of commuting vector fields on F. Note it might happen there are two different $E, E' \in \mathcal{G}_F$ with $G_E = G'_E$. Then X_E commutes with $X_{E'}$ by the previous Remark 25. Since $T_pH = T_pF' \cap T_pF$ and $T_p^+F = T_p^+P \cap T_pF$ for any $F' \in \mathcal{F}^1(P) \backslash \{F\}$ and $p \in H \in \mathcal{F}^1(F) \cap \mathcal{F}^1(F')$ we see X_E is normal to E in F. Finally, $\{X_E : E \in \mathcal{G}_F\}$ is a system of commuting normal vector fields on F.

Definition 35 (Collar Neighborhoods). *Let P be a manifold with embedded faces and $F \in \mathcal{F}^1(P)$. A* ***collar neighborhood of*** *F is an open neighborhood $C(F,\varepsilon) \subset P$ of F together with a diffeomorphism $\varphi_F : F \times [0,\varepsilon) \to C(F,\varepsilon)$ for some $\varepsilon > 0$ which satisfies $\varphi_F(p,0) = p$ for every $p \in F$.*

Two collar neighborhoods $C(F,\varepsilon)$, $C(F',\varepsilon)$ for $F, F' \in \mathcal{F}^1(P)$ are called ***compatible*** *if*

$$\varphi_{F'}(\varphi_F(p,t),s) = \varphi_F(\varphi_{F'}(p,s),t) \tag{19}$$

for every $(p,t,s) \in F \cap F' \times [0,\varepsilon) \times [0,\varepsilon)$.

A ***system of compatible collar neighborhoods*** *is an indexed set $C(\mathcal{G},\varepsilon) = \{C(G,\varepsilon) : G \in \mathcal{G}\}$ for some $\mathcal{G} \subset \mathcal{F}^1(P)$ and $\varepsilon > 0$ such that elements of $C(\mathcal{G},\varepsilon)$ are pairwise compatible.*

Let $C(\mathcal{G},\varepsilon)$ be a system of compatible collar neighborhoods. Let $n \in \mathbb{N}$, $\{G_1, \ldots, G_n\} \subset \mathcal{G}$ and $H \in \mathcal{F}^n(P)$, $H \subset G_1 \cap \ldots \cap G_n$. We define $\varphi_H : H \times [0,\varepsilon)^n \to P$ by

$$\varphi_H(p, t_1, \ldots, t_n) = \varphi_{G_n}(\ldots \varphi_{G_1}(p,t_1), \ldots), t_n)$$

for every $p \in H, t_i \in [0,\varepsilon)$. We also define $C(H;\varepsilon) = \mathrm{Im}(\varphi_H)$. The system of compatible collar neighborhoods is called ***neat*** *if for every $n \in \mathbb{N}$ and $\{G_1, \ldots, G_n\} \subset \mathcal{G}$ holds*

$$C(G_1,\varepsilon) \cap \ldots \cap C(G_n,\varepsilon) = \bigsqcup_{\substack{H \in \mathcal{F}^n(P) \\ H \subset G_1 \cap \ldots \cap G_n}} C(H,\varepsilon). \tag{20}$$

Remark 27 (Properties of Collar Neighborhoods). (i) Note that if $F' \in \mathcal{F}^1(P), F' \neq F$ then $\varphi_F(p,t) \in F'$ for all $p \in F \cap F'$ and $t \in [0,\varepsilon)$; therefore, equation 19 makes sense. Indeed, a diffeomorphism maps 1-faces onto 1-faces and φ_F restricts to the identity on the 1-face $F \times \{0\}$ of $F \times [0,\varepsilon)$.

(ii) If $C(G,\varepsilon)$ is a collar neighborhood of G in P and $H \in \mathcal{F}^1(G) \cap \mathcal{F}^1(F)$ for some 1-face F then we define

$$C_F(H,\varepsilon) = \varphi_G(H \times [0,\varepsilon)) \quad , \quad (\varphi_F)_H = \varphi_G\big|_{H \times [0,\varepsilon)}$$

This is a collar neighborhood of H in $F:$ Because $\varphi_G(H \times [0,\varepsilon)) \subset F$ from (i) we have $(\mathrm{d}\varphi_G)_p(T_pH \oplus \mathbb{R}) \subset T_{\varphi_G(p,t)}F$ for $p \in H$; hence, these vector spaces are equal from the fact that $(\mathrm{d}\varphi_G)_p$ is an isomorphism and from dimensional reasons. It follows from the Inverse Function Theorem that $C_F(H,\varepsilon) \subset F$ is open and $(\varphi_F)_H : H \times [0,\varepsilon) \to C_F(H,\varepsilon)$ is a diffeomorphism. Therefore, it is a collar neighborhood. In addition, it holds:

$$C(G,\varepsilon) \cap F = \bigsqcup_{H \in \mathcal{F}^1(G) \cap \mathcal{F}^1(F)} C_F(H,\varepsilon)$$

Indeed, the inclusion "$\subset$" into the union holds since φ_G maps faces to faces and the other inclusion is by definition of $C_F(H,\varepsilon)$. The union is disjoint since a point in $C_F(H,\varepsilon) \cap C_F(H',\varepsilon)$ has pre-image under φ_G in $H \cap H'$ but it holds either $H \cap H' = \emptyset$ or $H = H'$ for $H, H' \in \mathcal{F}^1(G) \cap \mathcal{F}^1(F)$. The construction above will be called *restriction of $C(G,\varepsilon)$ to F*. It is easy to see that restriction of a (neat) compatible system of collar neighborhoods $C(\mathcal{G},\varepsilon)$

on P to an $F \in \mathcal{F}^1(P)$ is a (neat) compatible system of collar neighborhoods on F.

(iii) If $H \in \mathcal{F}^n(P)$, $H \subset G_1 \cap \ldots \cap G_n$ then $C(H,\varepsilon) \subset P$ is open and $\varphi_H : H \times [0,\varepsilon)^n \to P$ is a diffeomorphism : We prove this by induction on $\dim(P)$. If $\dim(P) = 0$ then there is no face and nothing to prove. Let $\dim(P) > 0$ and let the claim hold for all smaller dimensions. By (ii) $C(\mathcal{G},\varepsilon)$ restricts to a system of compatible collar neighborhoods on G_1. Then $H \in \mathcal{F}^1(G_1)$ and we may write $\varphi_H(p, t_1, \ldots, t_n) = \varphi_{G_1}((\varphi_{G_1})_H(p, t_2, \ldots, t_n), t_1)$ for all $p \in H$, $t_i \in [0,\varepsilon)$ where $(\varphi_{G_1})_H : H \times [0,\varepsilon)^{n-1} \to G_1$. However, $(\varphi_{G_1})_H$ is a diffeomorphism onto the open subspace $C_{G_1}(H,\varepsilon) \subset G_1$ by the inductive assumption. In addition, φ_{G_1} is a diffeomorphism; hence, $\varphi_{G_1}(C_{G_1}(H,\varepsilon)) = C(H,\varepsilon) \subset P$ is open and φ_H a diffeomorphism.

(iv) It is clear that if $C(\mathcal{G},\varepsilon)$ is a (neat) system of compatible collar neighborhoods and $0 < \varepsilon' < \varepsilon$ then $C(\mathcal{G},\varepsilon')$ is a (neat) system of compatible collar neighborhoods again.

Lemma 32 (Weaker Neatness Condition). *Let P be a manifold with embedded faces and $C(\mathcal{G},\varepsilon)$ a system of compatible collar neighborhoods. Then $C(\mathcal{G},\varepsilon)$ is neat if and only if for every $G, G' \in \mathcal{G}$ holds*

$$C(G,\varepsilon) \cap C(G',\varepsilon) = \bigsqcup_{H \in \mathcal{F}^1(G) \cap \mathcal{F}^1(G')} C(H,\varepsilon)$$

Proof. One implication is trivially satisfied by definition. We will prove the other implication by induction on $n \geq 1$ in Formula 20. The case $n = 1$ holds trivially. Let $n > 1$ and assume the lemma holds for all smaller n. Note that in Formula 20 the inclusion "$\supset$" holds trivially and the union is disjoint automatically. Thus, it suffices to prove "$\subset$":

Let $p \in C(G_1,\varepsilon) \cap \ldots \cap C(G_n,\varepsilon)$. By the inductive assumption we get for every $i = 1, \ldots, n$ a point $q_i \in G_1 \cap \ldots \cap \hat{G}_i \cap \ldots \cap G_n$ and a vector

$\bar{t}^i = (t_1^i, \ldots, \hat{t}_i^i, \ldots, t_n^i) \in [0, \varepsilon)^{n-1}$ such that if we apply $\varphi_{G_j}(-, t_j^i)$ for all $j \neq i$ successively to q_i we get p. If $i, i' \in \{1, \ldots, n\}$ are two different indices we use compatibility of collar neighborhoods to group together maps φ_{G_j}, $j \neq i, i'$ which are applied to q_i and $q_{i'}$ in both cases i and i', respectively. We use successively the fact that these maps are one-to-one and get that : $t_j^i = t_j^{i'}$ for $j \neq i, i'$ and $\varphi_{G_{i'}}(q_i, t_{i'}^i) = \varphi_{G_i}(q_{i'}, t_i^{i'})$. First of all, we see we may denote $t_j = t_j^i$ for every $i \neq j$. We also denote $\bar{t} = (t_1, \ldots, t_n)$. Secondly, Equation 32 which we assume implies there is a $z_{ii'} \in G_i \cap G_{i'}$ such that $q_i = \varphi_{G_{i'}}(z_{ii'}, t_{i'})$ and $q_{i'} = \varphi_{G_i}(z_{ii'}, t_i)$. We use the fact that $\varphi_{G_{i'}}$ is a diffeomorphism and maps faces to faces to see that $z_{ii'} \in G_1 \cap \ldots \cap G_n$ since q_i lies in all G_j, $j \neq i$ and $q_{i'}$ in all G_j, $j \neq i'$ and $i \neq i'$. Therefore, we have points $z_{ii'} \subset G_1 \cap \ldots \cap G_n$ for all $i \neq i'$ which are mapped to the same point p by the one-to-one composite map $\varphi_{G_1} \circ \ldots \circ \varphi_{G_n}$ where $\bar{t}$ is fixed; hence, there is a $H \in \mathcal{F}^n(P)$, $H \subset G_1 \cap \ldots \cap G_n$ and $z \in H$ such that $z = z_{ii'} \in H$ for all $i \neq i'$. Therefore, the lemma is proven as $\varphi_H(z, \bar{t}) = p$. $\square$

Lemma 33 (Shrinking to a Neat System of Collar Neighborhoods). *Let P be a compact manifold with embedded faces. Let $C(\mathcal{G}, \varepsilon)$ be a system of compatible collar neighborhoods. Then there exists an $0 < \varepsilon' < \varepsilon$ such that the system $C(\mathcal{G}, \varepsilon')$ is neat.*

Proof. We prove the lemma by induction on $|\mathcal{G}|$: If $|\mathcal{G}| = 0$ then there is nothing to prove. Let $F \in \mathcal{G}$. Assume $C(\mathcal{G} \backslash \{F\}, \varepsilon)$ is a neat system of compatible collar neighborhoods. According to Lemma 32 it suffices to check there is an $0 < \varepsilon' < \varepsilon$ such that $C(G, \varepsilon') \cap C(F, \varepsilon') = \bigsqcup_{H \in \mathcal{F}^1(G) \cap \mathcal{F}^1(F)} C(H, \varepsilon')$ for every $G \in \mathcal{G} \backslash \{F\}$. Relations between the other element of $\mathcal{G}$ remain satisfied after shrinking ε by (iv) of Remark 27.

It suffices to show "$\subset$" since the rest holds. To get a contradicition, suppose that for every $n \in \mathbb{N}$ there is an $0 < \varepsilon_n < \frac{\varepsilon}{n}$ and a $p_n \in C(F, \varepsilon_n)$

such that there is a $G \in \mathcal{G}$ with $p_n \in C(F, \varepsilon_n) \cap C(G, \varepsilon_n)$ but there is no $H \in \mathcal{F}^1(G) \cap \mathcal{F}^1(F)$ containing p_n. We may write $p_n = \varphi_F(q_n, t_n)$ for some $(q_n, t_n) \in F \times [0, \frac{\varepsilon_F}{n})$. Because F is compact we may assume $q_n \mapsto q \in F$. Therefore, $(q_n, t_n) \mapsto (q, 0)$ and $p_n \mapsto p$ for some $p \in F$. Because $\mathcal{G}$ is finite there is a $G \in \mathcal{G} \backslash \{F\}$ such that $p_n \in C(G, \varepsilon_n)$ for infinitely many $n \in \mathbb{N}$. Similarly as in the case of F we deduce $p_n \mapsto p' \in G$ which equals p since P is Hausdorff. Therefore, $p \in F \cap G$ and there is an $H \in \mathcal{F}^1(G) \cap \mathcal{F}^1(F)$ containing p. Therefore, $p_n \in C(H, \varepsilon)$ for n big enough because it is a neighborhood of p. We get a contradiction with the choice of p_n as $p_n \in C(H, \varepsilon) \cap C(F, \varepsilon_n) \cap C(G, \varepsilon_n) = C(H, \varepsilon_n)$. □

LEMMA 34 (Collar Neighborhoods Generated by Vector Fields). *Let P be a compact connected manifold with embedded faces, $\mathcal{G} \subset \mathcal{F}^1(P)$ and let $\{X_G : G \in \mathcal{G}\}$ be a system of commuting normal vector fields. Then the flows of $\{X_G : G \in \mathcal{G}\}$ generate a neat system of compatible collar neighborhoods $C(\mathcal{G}, \varepsilon)$ for some $\varepsilon > 0$.*

Proof. First of all, we will check that a single normal vector field $X = X_G : U \to TU$, $U = U_G$ generates a collar neighborhood $C(G, \varepsilon)$:

By (c) of Theorem 15 for every $p \in G$ there is an $\varepsilon_p > 0$ and an open neighborhood $U_p \subset U$ of p such that $U_p \times [0, \varepsilon_p) \subset W$. Because G is compact we may pick a finite cover $\{U_{p_n} : n = 1, \ldots, N\} \subset \{U_p : p \in G\}$ of G. We shrink U to $\bigcup_{n=1}^N U_{p_n}$ and denote $\varepsilon = \min\{\varepsilon_{p_n} : n = 1, \ldots, N\}$. Therefore, we get $U \times [0, \varepsilon) \subset W$. We define $\varphi = \psi\big|_{G \times [0,\varepsilon)}$ and check the conditions of the Inverse Function Theorem 20 for φ near G : We have $\varphi(G \times \{0\}) = G \subset \partial U$ as $\psi^0 = \mathrm{id}$ and $\varphi(\partial G \times [0, \varepsilon)) \subset \partial U$ since X is normal to G. Therefore, $\varphi(\partial(G \times [0, \varepsilon))) \subset \partial U$. As for the condition on differential, let $p \in G$. We apply the chain rule to $\varphi : G \times [0, \varepsilon) \hookrightarrow U \times [0, \varepsilon) \xrightarrow{\psi} P$ to see $(\mathrm{d}\varphi)_{(p,t)} = (\mathrm{d}\psi^t)_p \times 0 + 0 \times (\mathrm{d}\psi_p)_t$ on $T_{(p,t)}(G \times [0, \varepsilon))$. We calculate

$(\mathrm{d}\psi^0)_p(T_pG) = T_pG$ as $\psi^0 = \mathrm{id}_U$ and $(\mathrm{d}\psi_p)_0(1) = X_p$ where 1 is a non-zero element of $T_0[0,\varepsilon) \simeq \mathbb{R}$. Since X is inward pointing on G we have $X_p \in T_p^+U \backslash T_pG$; hence, $\mathrm{Im}(\mathrm{d}\varphi_{(p,0)}) = T_pG \oplus \mathrm{span}\{X(p)\} = T_pU$. Therefore, $\mathrm{d}\varphi_{(p,0)}$ is an isomorphism by dimensional reasons. Now we can apply the IFT 20 to φ at $(p,0)$ and get an open neighborhood $V_p \subset G \times [0,\varepsilon)$ of $(p,0)$ such that $\varphi(V_p) \subset U$ is open and $\varphi : V_p \to \varphi(V_p)$ is a diffeomorphism of open submanifolds. We will check there is an $0 < \varepsilon' < \varepsilon$ such that φ is injective on $G \times [0,\varepsilon')$. For the contrary, suppose for every $n \in \mathbb{N}$ there is a pair of distinct points $(p_n,t_n),(q_n,s_n) \in G \times [0,\frac{\varepsilon}{n})$ such that $\varphi(p_n,t_n) = \varphi(q_n,s_n)$. We may assume $p_n \mapsto p, q_n \mapsto q$ for some $p,q \in G$ because G is compact. Since φ is continuous we have $p = \varphi(p,0) = \varphi(q,0) = q$. Therefore, $(p_n,t_n),(q_n,t_n) \in V_p$ for some n big enough. Nevertheless, this is a contradiction with injectivity of $\varphi\big|_{V_p}$. Therefore, $\varphi : G \times [0,\varepsilon') \to P$ is an injective local diffeomorphism. Therefore, it is a diffeomorphism onto its open image $C(G,\varepsilon')$ and we may just rename $\varepsilon' \mapsto \varepsilon$ to get the desired.

We apply the same procedure to X_G and get $C(G,\varepsilon_G)$ for some $\varepsilon_G > 0$ for every $G \in \mathcal{G}$. Therefore, we end up with a system of (not necessary compatible) collar neighborhoods $C(\mathcal{G},\varepsilon)$ where $\varepsilon = \min\{\varepsilon_G : G \in \mathcal{G}\}$. Let $G, G' \in \mathcal{G}$ be a pair of distinct 1-faces. We apply Lemma 31 to $X_G, X_{G'}$ at every $p \in G \cap G'$ to get neighborhoods $U_p \subset U_G \cap U_{G'}$ and $\varepsilon_p > 0$ such that the flows $\psi_G, \psi_{G'}$ commute for $(q,t,s) \in U_p \times [0,\varepsilon_p) \times [0,\varepsilon_p)$. This means, in particular, that $\varphi_{G'}(\varphi_G(q,t),s) = \varphi_G(\varphi_G(q,s),t)$ when $q \in U_p \cap G \cap G'$. Because $G \cap G'$ is compact, as a closed subspace of the compact space G, we may pick a finite subcover of $\{U_p : p \in G \cap G'\}$ and find an $0 < \varepsilon' < \varepsilon$ such that φ_G, $\varphi_{G'}$ commute for $(q,t,s) \in G \cap G' \times [0,\varepsilon') \times [0,\varepsilon')$. Therefore, when we shrink $\varepsilon \mapsto \varepsilon'$ we get that $C(G,\varepsilon)$ and $C(G',\varepsilon)$ are compatible. We do the same procedure for every pair $G, G' \in \mathcal{G}$ and always shrink ε so that the chosen pair of collar neighborhoods becomes compatible. The pairs which were already compatible remain compatible after shrinking of ε. Since $\mathcal{G}$ is finite we end up after a finite number of steps with a compatible system of collar neighborhoods $C(\mathcal{G}, \varepsilon)$. Lemma 33 about shrinking to a neat system finishes the proof. $\square$

REMARK 28. Note that taking restriction of a system of normal vector fields $\{X_G : G \in \mathcal{G}\}$ by Remark 26 commutes with taking restrictions of compatible system of collar neighborhoods generated by $\{X_G : G \in \mathcal{G}\}$ according to (ii) of Remark 27. This is easy to see from what was proven in these remarks.

LEMMA 35 (Extending Commuting Normal Vector Fields). *Let P be a compact manifold with embedded faces, $\mathcal{G} \subset \mathcal{F}^1(P)$ and let $\{X_G : G \in \mathcal{G}\}$ be a system of commuting normal vector fields on P. Let $F \in \mathcal{F}^1(P)\backslash\mathcal{G}$ and let $\mathcal{H} \subset \mathcal{G}_F$ be a subset of the boundary of F in $\mathcal{G}$. Suppose that the following conditions are satisfied:*

(i) For every $H \in \mathcal{H}$ there is a an open neighborhood $U_H \subset G_H$ of H and vector field $X_H : U_H \to TU_H$ normal to H in G_H.

(ii) $\left[X_H, X_G\big|_{G_H}\right] = 0$ on $U_G \cap U_H$ for every $H \in \mathcal{H}$ and $G \in \mathcal{G}\backslash\{G_H\}$.

(iii) $X_H = X_{H'}$ on $U_H \cap U_{H'}$ for every $H, H' \in \mathcal{H}$

Then there is an open neighborhood $U_F \subset P$ of F and a vector field $X_F : U_F \to TU_F$ such that

(i) $\{X_G : G \in \mathcal{G}\} \cup \{X_F\}$ is a commuting system of normal vector fields.

(ii) $X_F\big|_{G_H} = X_H$ on the common domain for every $H \in \mathcal{H}$.

after possible shrinking of domains of all the vector fields we work with.

Proof. We will prove this statement by induction on $\dim(P)$: If $\dim(P) = 0$ there are no faces and there is nothing to prove. Assume $\dim(P) > 0$ and that the statement holds for all lower dimensions. We distinguish the following cases :

If $\mathcal{G} = \emptyset$ then automatically $\mathcal{H} = \emptyset$ and we just have to construct a normal vector field to F with no additional conditions to be satisfied. The construction proceeds as follows : Let $\{(U_n, \varphi_n) : n \in \mathbb{N}\}$ be a regular cover of P and $\{\lambda_n : P \to [0,1]; n \in \mathbb{N}\}$ an associated partition of unity from Theorem 17. Let $n \in \mathbb{N}$. If $F \cap U_n = \emptyset$ we let $v_n : U_n \to TU_n$ to be the zero vector field. If $F \cap U_n \neq \emptyset$ and $\varphi_n(F \cap U_n) \subset \partial_i \mathbb{R}_+^{\dim(P)}$ for some i we define $v_n : U_n \to TP$ by the formula

$$v_n(p) = (\mathrm{d}\varphi_n)_p^{-1}(0, \ldots, 0, 1, 0, \ldots, 0)$$

for $p \in U_n$ where 1 is on the i-th position. Since v_n is normal to F in coordinates it is normal to F on the manifold. Altogether, v_n is normal to F for every $n \in \mathbb{N}$. We define a smooth vector field $X_F : P \to TP$ by the formula

$$X_F(p) = \sum_{n=1}^{\infty} \lambda_n(p) v_n(p)$$

for every $p \in P$. At a point $p \in P$ it is a finite linear combination of vectors normal to F. Therefore, it is a vector field normal to F and the construction is finished.

Let $\mathcal{G} \neq \emptyset$ and suppose $\mathcal{H} = \mathcal{G}_F$ is the entire boundary of F in $\mathcal{G}$. We are in the situation that X_F is prescribed as X_H in ∂P along the boundary of F in $\mathcal{G}$ (see Figure 6 below) and we want to extend it to a vector field $\tilde{X}$ on a neighborhood of the boundary of F in $\mathcal{G}$ in P. For this purpose, we construct convenient systems of neighborhoods on P and likewise on G for $G \in \mathcal{G}$ as follows :

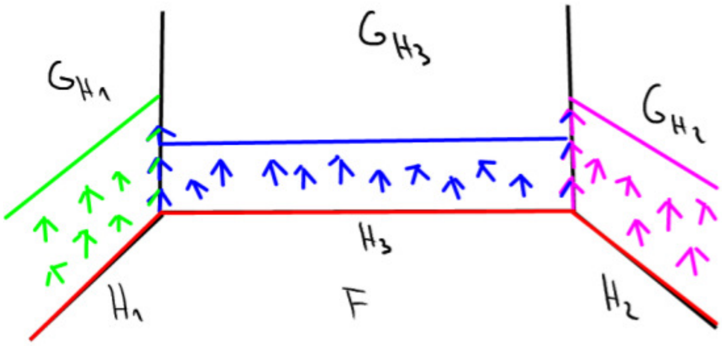

Figure 6: The situation $\mathcal{H} = \mathcal{G}_F$. The vector field X_F is prescribed near H_1, H_2, H_3 on faces $G_{H_1}, G_{H_2}, G_{H_3}$, respectively.

We denote $\mathcal{G}' = \mathcal{G} \cup \{F\}$. Let $G \in \mathcal{G}$ and consider the boundary $\mathcal{G}'_G$ of G in $\mathcal{G}' = \mathcal{G} \cup \{F\}$. Note that $\mathcal{G}'_G \backslash \mathcal{H} = \mathcal{G}_G$. We define a system of vector fields $\{Y_E : E \in \mathcal{G}'_G\}$ on G as follows : If $H \in \mathcal{H} \cap \mathcal{G}'_G$ we set $Y_E = X_E$. Otherwise, we define $\{Y_E : E \in \mathcal{G}_G\}$ as the restriction of $\{X_G : G \in \mathcal{G}\}$ to G. By Remark 26 this restriction is a system of commuting normal vector fields on G. Assumptions (i) to (iii) guarantee $\{X_H : H \in \mathcal{H} \cap \mathcal{G}'_G\}$ is a system of commuting normal vector fields on G which commute pairwise with all the restrictions. Therefore, the entire set $\{Y_E : E \in \mathcal{G}'_G\}$ is a commuting system of normal vector fields on G. For every $G \in \mathcal{G}$ we apply Lemma 34 and get a neat system of compatible collar neighborhoods $C_G(\mathcal{G}'_G, \varepsilon_G)$ on G generated by $\{Y_E : E \in \mathcal{G}'_G\}$. The same lemma gives a neat system of compatible collar neighborhoods $C(\mathcal{G}, \varepsilon')$ on P generated by $\{X_G : G \in \mathcal{G}\}$. We set $\tilde{\varepsilon} = \min\{\varepsilon_G : G \in \mathcal{G}\} \cup \{\varepsilon'\}$ and take an $0 < \varepsilon < \tilde{\varepsilon}$. We abbreviate $C(\mathcal{G}) = C(\mathcal{G}, \varepsilon)$ and $C_G(\mathcal{G}'_G) = C(\mathcal{G}'_G, \varepsilon)$. These are neat systems of compatible collar neighborhoods as shrinking of ε does not change this property. Note that by Remark 28 for any $G, G' \in \mathcal{G}$ it holds $C(G) \cap G' = \bigsqcup_{E \in \mathcal{F}^1(G) \cap \mathcal{F}^1(G')} C_{G'}(E)$ and $(\varphi_{G'})_E = \varphi_G\big|_E$ for every $E \in \mathcal{F}^1(G) \cap \mathcal{F}^1(G')$. Therefore, we may only work with maps $\{\varphi_G : G \times [0,\varepsilon) \to P; G \in \mathcal{G}\}$ and $\{\varphi_H : H \times [0,\varepsilon) \to G_H; H \in \mathcal{H}\}$ which are diffeomorphisms onto their images and whose fixed-time maps commute among themselves on overlaps of their domains. The construction of convenient neighborhoods is thereby completed.

Now we can extend X_H for some $H \in \mathcal{G}_F$ as follows: Because $C_{G_H}(H) \subset U_H$ by construction we may define $\tilde{U}_H = \varphi_{G_H}(C_{G_H}(H) \times [0,\varepsilon)) \subset C(G_H)$ and a vector field $\tilde{X}_H : \tilde{U}_H \to T\tilde{U}_H$ by the formula

$$\tilde{X}_H(\varphi_{G_H}(r,t)) = \mathrm{d}(\varphi^t_{G_H})_r X_H(r) \tag{21}$$

for all $(r,t) \in C_{G_H}(H) \times [0,\varepsilon)$. The $\tilde{U}_H \subset P$ is open since collar neighborhoods are open subsets and $\tilde{X}_H$ is smooth since φ_{G_H} is a diffeomorphism. The following construction (depicted in Figure 7 below) will be used further:

Let $p \in \tilde{U}_H \cap C(G)$ for some $G \in \mathcal{G}$. Then $p \in C(G) \cap C(G_H)$ and neatness of $C(\mathcal{G})$ implies there is an $E \in \mathcal{F}^1(G) \cap \mathcal{F}^1(G_H)$, $q \in E$, $t_0, s_0 \in [0,\varepsilon)$ such that $p = \varphi_G(\varphi_{G_H}(q,s_0),t_0) = \varphi_{G_H}(\varphi_G(q,t_0),s_0)$. By definition of $\tilde{U}_H$ and the fact that φ_{G_H} is one-to-one we have $r = \varphi_G(q,t_0) \in C_{G_H}(H)$. Since $r \in C_{G_H}(H) \cap C_{G_H}(E)$ neatness of $C(\mathcal{G}_H)$ gives a $z \in H \cap E$ and $u_0, v_0 \in [0,\varepsilon)$ such that $r = \varphi_G(\varphi_H(z,u_0),v_0) = \varphi_H(\varphi_G(z,v_0),u_0)$. In particular, $q = \varphi_H(z,u_0)$ and $v_0 = t_0$ since φ_G is one-to-one. When we fix u_0 we see from these commutation relations that $\varphi_G(q,t) = \varphi_H(\varphi_G(z,t),u_0)$ for all $t \in [0,\varepsilon)$. Therefore, the curve $(\varphi_G)_q : [0,\varepsilon) \to G_H$ lies entirely in $C_{G_H}(H)$.

Now we will show that $\tilde{X}_H$ commutes with X_G on $\tilde{U}_H \cap C(G)$ for every $G \in \mathcal{G}$: Let $p \in \tilde{U}_H \cap C(G)$ for some $G \in \mathcal{G}$. Then referring to the paragraph above there are $q \in G \cap G_H$, $t_0, s_0 \in [0,\varepsilon)$, $r = \varphi_G(q,t_0)$ such that $p = \varphi_{G_H}(r,s_0)$ and $(\varphi_G)_q : [0,\varepsilon) \to G_H$ lies in $C_{G_H}(H)$. In particular, for small t such that $t + t_0 \in [0,\varepsilon)$ we have $\psi_G(r,t) = \psi_G(\varphi_G(q,t_0),t) = \varphi_G(q,t_0+t) \in C_{G_H}(H)$ by the group law. Once more by the group law we calculate $\psi_G(p,t) = \psi_G(\varphi_{G_H}(r,s_0),t) = \psi_G(\varphi_{G_H}(\varphi_G(q,t_0),s_0),t) = \psi_G(\varphi_G(\varphi_{G_H}(q,s_0),t_0),t) =$

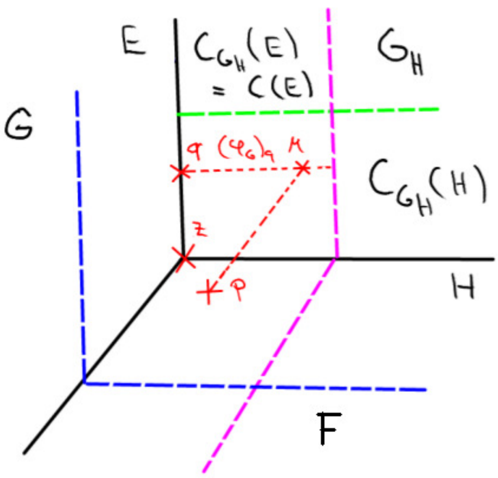

Figure 7: The situation and notation near the overlap of two 1-faces $G, G_H \in \mathcal{G}$. The dotted lines in space bound $C(G)$ and $C(G_H)$.

$\varphi_G(\varphi_{G_H}(q, s_0), t_0 + t) = \varphi_{G_H}(\varphi_G(q, t_0 + t), s_0) = \varphi_{G_H}(\psi_G(\varphi_G(q, t_0), t), s_0) = \varphi_{G_H}(\psi_G(r, t), s_0)$. Now we may rewrite the definition of $\tilde{X}_H$ at $\psi_G^t(p)$ for small t as

$$(\tilde{X}_H)_{\psi_G^t(p)} = (\mathrm{d}\varphi_{G_H}^{s_0})_{\psi_G^t(r)}(X_H)_{\psi_G^t(r)} \tag{22}$$

We can now make the following, so far just formal calculation

$$(\mathcal{L}_{X_G}\tilde{X}_H)_p = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\mathrm{d}\psi_G^t)_p^{-1}(\tilde{X}_H)_{\psi_G^t(p)} =$$
$$= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\mathrm{d}\psi_G^t)_p^{-1}(\mathrm{d}\varphi_{G_H}^{s_0})_{\varphi_G^t(r)}(X_H)_{\psi_G^t(r)} = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\mathrm{d}\varphi_{G_H}^{s_0})_r(\mathrm{d}\psi_G^t)_r^{-1}(X_H)_{\psi_G^t(r)} =$$
$$= (\mathrm{d}\varphi_{G_H}^{s_0})_r(\mathcal{L}_{X_G}X_H)_r = 0$$

where on the last line it is used that X_H commutes with restriction of X_G to G_H. The only thing which remains to be justified is the swap of differentials on the middle line : Consider the equation

$$\psi_G^t \circ \varphi_{G_H}^{s_0} = \varphi_{G_H}^{s_0} \circ \psi_G^t \tag{23}$$

We claim it holds at least on an open neighborhood of r in G_H for every small enough non-negative t and can be differentiated using the chain rule: By (c) of Theorem 15 there exist open neighborhoods $U_p, U_r \subset C(G) \cap C(G_H)$ of p, r and $0 < \delta < \varepsilon$ such that $U_p \cup U_r \times [0, \delta) \subset W_G$ where both intervals can be taken right-open since we are within $C(G)$. We may take $t \in [0, \delta)$. Now we just shrink U_r so that $\varphi^{s_0}_{G_H}(U_r \cap G_H) \subset U_p$. Since ψ^t_G preserves G_H we have $\psi^t_G(U_r \cap G_H) \subset G_H$. Therefore, the equation holds on $U_r \cap G_H$ and we can differentiate. We apply the chain rule to the composition of maps $\varphi^{s_0}_{G_H} : U_r \cap G_H \to U_p$ and $\psi^t : U_p \to P$ on the left hand side and to the composition of $\psi^t_G\big|_{U_r \cap G_H} : U_r \cap G_H \to G_H$ and $\varphi^{s_0} : G_H \to P$ on the right hand side. Using the fact that the differential of $\psi^t_G\big|_{U_r \cap G_H}$ is the same as restriction of $\mathrm{d}\psi^t_G$ to TG_H since ψ^t_G preserves G_H we get

$$(\mathrm{d}\psi^t_G)_p(\mathrm{d}\varphi^{s_0}_{G_H})_r = (\mathrm{d}\varphi^{s_0}_{G_H})_{\psi^t_G(r)}(\mathrm{d}\psi^t_G)_r \quad \text{on } T_rG_H \tag{24}$$

For small t both $(\mathrm{d}\psi^t_G)_p$ / $(\mathrm{d}\psi^t_G)_r$ are invertible and we may apply the inverses to the equation from the left / right to get

$$(\mathrm{d}\varphi^{s_0}_{G_H})_r(\mathrm{d}\psi^t_G)^{-1}_r = (\mathrm{d}\psi^t_G)^{-1}_p(\mathrm{d}\varphi^{s_0}_{G_H})_{\psi^t_G(r)} \quad \text{on } (\mathrm{d}\psi^t_G)_r(T_rG_H) \tag{25}$$

where $(\mathrm{d}\psi^t_G)_r(T_rG_H) = T_{\psi^t_G(r)}G_H$ because ψ_G preserves G_H. Because of $X_{\psi^t_G(r)} \in T_{\psi^t_G(r)}G_H$ the swap of differentials for small non-negative t is indeed legitimate. Therefore, $[\tilde{X}_H, X_G] = 0$ on $\tilde{U}_H \cap C(G)$ is proven.

Now we will check that $\tilde{X}_H$ is normal to F on $\tilde{U}_H$: Let $p \in \tilde{U}_H \cap G$ for some $G \in \mathcal{F}^1(P)$. We have to show that $(\tilde{X})_p \in T_pG$ if $G \neq F$ and $(\tilde{X})_p \in T^+_pP \backslash T_pF$ if $F = G$. If $G = G_H$ then $s_0 = 0$, $p = r = q$ and $(\tilde{X}_H)_p = (X_H)_p \in T_pG_H$. Let $G \neq G_H$. In the notation of the previous paragraphs we have $t_0 = 0$, $q = r \in E \cap C_{G_H}(H)$, $E \in \mathcal{F}^2(P)$, $E \subset G \cap G_H$ and $p = \varphi_{G_H}(q, s_0)$. Since $\varphi_{G_H}(E \times [0, \varepsilon)) \subset G$ we have $(\mathrm{d}\varphi^{s_0}_{G_H})_r(T_rE) \subset T_pG$. If $G \neq F$ then $E \neq H$ because F is the unique 1-face containing H and different from G_H. Therefore, $(X_H)_r \in T_rE$ since X_H is normal to H in G_H; hence, $(\tilde{X}_H)_p = (\mathrm{d}\varphi^{s_0}_{G_H})_r(X_H)_r \in T_pG$ which is what we want. Finally, assume $G = F$. First of all, there is a $q \in E \cap C_{G_H}(H) \subset C_{G_H}(E) \cap C_{G_H}(H)$ and neatness of $C(\mathcal{G}'_{G_H})$ implies $H \cap E \neq \emptyset$. Because $E, H \in \mathcal{F}^1(G_H) \cap \mathcal{F}^1(F)$ we must necessarily have $E = H$. Therefore, we may suppose $(X_H)_r \in T^+_r G_H \backslash T_r H$. By Lemma 8 we have $(\mathrm{d}\varphi^{s_0}_{G_H})_r(T^+_r G_H) \subset T^+_p P$; hence, $(\tilde{X}_H)_p \in T^+_p P$. It remains to show that $(\tilde{X}_H)_p \notin T_pF$. We have

$$(\mathrm{d}\varphi_{G_H})_{(r,s_0)}(T_rG_H \oplus \mathbb{R}) = (\mathrm{d}\varphi^{s_0}_{G_H})_r(T_rG_H) \oplus \operatorname{span}\{(X_{G_H})_p\} = T_pP$$

as in the proof of Lemma 34. Because X_{G_H} is normal to $G_H \neq F$ we have $\operatorname{span}\{(X_{G_H})_p\} \subset T_pF$. We also have $T_rH \subset T_rF \cap T_rG_H$; thus, $(\mathrm{d}\varphi^{s_0}_{G_H})_r(T_rH) \subset T_pF$ as φ_{G_H} preserves faces but $(X_{G_H})_p \notin (\mathrm{d}\varphi^{s_0}_{G_H})_r(T_rH)$ as $(\mathrm{d}\varphi^{s_0}_{G_H})_r(T_rH) \subset (\mathrm{d}\varphi^{s_0}_{G_H})_r(T_rG_H)$. Because $(\mathrm{d}\varphi^{s_0}_{G_H})_r$ is injective it follows from dimensional reasons that

$$T_pF = (\mathrm{d}\varphi^{s_0}_{G_H})_r(T_rH) \oplus \operatorname{span}\{(X_{G_H})_p\}$$

For the contrary suppose $(\tilde{X}_H)_p \in T_pF$. Because $(\tilde{X}_H)_p \in (\mathrm{d}\varphi^{s_0}_{G_H})_r(T_rG_H)$ has empty intersection with $\operatorname{span}\{(X_{G_H})_p\}$ we must necessarily have $(\tilde{X}_H)_p \in (\mathrm{d}\varphi^{s_0}_{G_H})_r(T_rH)$. Since $(\mathrm{d}\varphi^{s_0}_{G_H})_r$ is injective we get $(X_H)_r \in T_rH$ which is a contradiction. Therefore, $(\tilde{X}_H)_p \neq T_pF$ and the proof that $\tilde{X}_H$ is normal to F is finished.

Next we will show that $\{\tilde{X}_H : H \in \mathcal{H}\}$ overlap on common domains: Let $H, H' \in \mathcal{H}$ and $p \in \tilde{U}_H \cap \tilde{U}_{H'}$. Using the notation from previous paragraphs again we have a $q \in G_H \cap G_{H'}$, $t_0, s_0 \in [0, \varepsilon)$ such that $p = \varphi_{G_H}(\varphi_{G_{H'}}(q, t_0), s_0) = \varphi_{G_{H'}}(\varphi_{G_H}(q, s_0), t_0)$ and the curves $(\varphi_{G_{H'}})_q : [0, \varepsilon) \to G_H$ / $(\varphi_{G_H})_q : [0, \varepsilon) \to G_{H'}$ lie in $C_{G_H}(H)$ / $C_{G_{H'}}(H')$. In particular, they are contained in the intersection of domains of $X_{G_{H'}}\big|_{G_H}$ and X_H / $X_{G_H}\big|_{G_{H'}}$ and $X_{H'}$ where these pairs of vector fields commute by assumption. Collar neighborhoods have been constructed as in the proof of theorem 34. Therefore, there are open neighborhoods U_{G_H} / $U_{G_{H'}}$ of G_H / $G_{H'}$ in P such that $U_{G_H} \times [0, \varepsilon) \subset W_{G_H}$, $U_{G_{H'}} \times [0, \varepsilon) \subset W_{G_{H'}}$. Remark 24 then asserts $(\mathrm{d}\psi^u_{G_H})_q$ / $(\mathrm{d}\psi^v_{G_{H'}})_q$ exist, are smooth in u / v, and are invertible. Therefore, we may use Equation 18, namely $\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}\psi^t)^{-1}_r Y_{\psi(r,t)} = (\mathrm{d}\psi^t)^{-1}_r (\mathcal{L}_X Y)_{\psi(r,t)}$, for all $t \in [0, \varepsilon)$ with $X = X_{G_{H'}}$, $\psi = \psi_{G_{H'}}$, $Y = X_H$ / $X = X_{G_H}$, $\psi = \psi_{G_H}$, $Y = X_{H'}$ and deduce

$$\begin{aligned} (X_H)_{\varphi_{G_{H'}}(q,v)} &= (\mathrm{d}\psi^v_{G_{H'}})_q (X_H)_q \\ (X_{H'})_{\varphi_{G_H}(q,u)} &= (\mathrm{d}\psi^u_{G_H})_q (X_{H'})_q \end{aligned} \tag{26}$$

for all $u, v \in [0, \varepsilon)$ because the vector fields commute. We also have $\mathrm{d}\varphi^u_{G_H} = \mathrm{d}\psi^u_{G_H}$ / $\mathrm{d}\varphi^v_{G_{H'}} = \mathrm{d}\psi^u_{G_{H'}}$ on TG_H / $TG_{H'}$; hence, the equations above hold when ψ is replaced by φ on the right-hand side because $(X_H)_q = (X_{H'})_q \in T_q(G_H \cap G_{H'})$ by assumption. We can now calculate

$$\begin{aligned} (\tilde{X}_H)_p = \mathrm{d}(\varphi^t_{G_H})_{\varphi_{G_{H'}}(q,t')} (X_H)_{\varphi_{G_{H'}}(q,t')} = \mathrm{d}(\varphi^t_{G_H})_{\varphi_{G_{H'}}(q,t')} \mathrm{d}(\varphi^{t'}_{G_{H'}})_q (X_H)_q = \\ = \mathrm{d}(\varphi^{t'}_{G_{H'}})_{\varphi_{G_H}(q,t)} \mathrm{d}(\varphi^t_{G_H})_q (X_{H'})_q = \mathrm{d}(\varphi^{t'}_{G_{H'}})_{\varphi_{G_H}(q,t)} (X_{H'})_{\varphi_{G_H}(q,t)} = (\tilde{X}_H)_p. \end{aligned}$$

using the relation $\mathrm{d}\varphi^t_{G_H} \circ \mathrm{d}\varphi^{t'}_{G_{H'}} = \mathrm{d}\varphi^{t'}_{G_{H'}} \circ \mathrm{d}\varphi^t_{G_H}$ valid on $T(G_H \cap G_{H'})$ as $\varphi^t_{G_H} \circ \varphi^{t'}_{G_{H'}} = \varphi^t_{G_H} \circ \varphi^{t'}_{G_{H'}}$ on $G_H \cap G_{H'}$ for all $t, t' \in [0, \varepsilon)$. Therefore, we have just proven that $\tilde{X}_H = \tilde{X}_{H'}$ on $\tilde{U}_H \cap \tilde{U}_{H'}$.

Finally, we may set $\tilde{U} = \bigcup_{H \in \mathcal{H}} \tilde{U}_H$ and define a smooth vector field $\tilde{X} : \tilde{U} \to T\tilde{U}$ by setting $\tilde{X}(p) = \tilde{X}_H(p)$ if $p \in \tilde{U}_H$ for $p \in \tilde{U}$ (See Figure 8)The previous paragraph shows this is a well defined vector field which is smooth since it locally equals $\tilde{X}_H$ for some H. From the same reason it is normal to F on $\tilde{U}$ and commutes with $\{X_G : G \in \mathcal{G}\}$ on $\tilde{U} \cap \bigcup C(\mathcal{G})$. Now we find a vector field X normal to F as in the first paragraph of the proof. It generates a collar neighborhood $C(F, \eta)$ of F in P for some $\eta > 0$. By taking

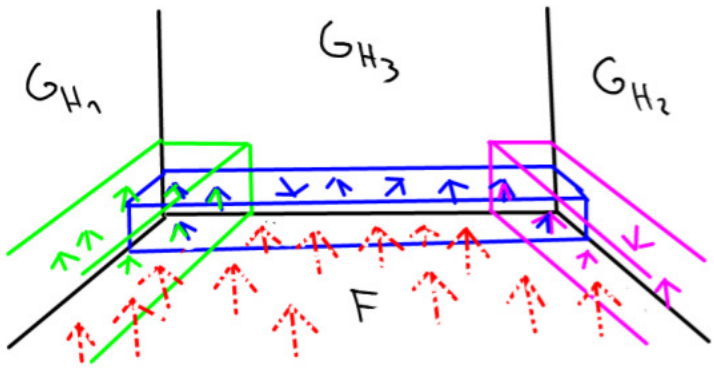

Figure 8: The situation $\mathcal{H} = \mathcal{G}_F$ after extending vector fields from $G_{H_1}, G_{H_2}, G_{H_3}$ to "boxes" along the boundary of F in $\mathcal{G}$ where they overlap and define a vector field $\tilde{X}$. The dotted vector field on a neighborhood of F in P is then constructed and glued together with $\tilde{X}$ to finally obtain X_F.

η small enough we may achieve $C(F,\eta) \cap \bigcup C(\mathcal{G}) \subset \tilde{U}$: Assume it is not possible. Then for every $n \in \mathbb{N}$ there is a $p_n \in C(F, \frac{\eta}{n})$ such that $p_n \in C(G)$ for some $G \in \mathcal{G}$ but $p_n \notin \tilde{U}$. We deduce $p_n \mapsto p \in F$ by compactness of F. Because $\mathcal{G}$ is finite there is a face $G \in \mathcal{G}$ such that $p_n \in C(G)$ for infinitely many $n \in \mathbb{N}$. Therefore, p is a limit point of $C(G)$. However, $\overline{C(G)} \subset C(G, \tilde{\varepsilon})$ by the choice of $0 < \varepsilon < \tilde{\varepsilon}$ and it follows $p \in F \cap C(G, \tilde{\varepsilon})$. Therefore, there is a $q \in G$ and $t \in [0, \tilde{\varepsilon})$ such that $p = \varphi_G(q,t)$. We have $q \in H$ for some $H \in \mathcal{F}^1(G) \cap \mathcal{F}^1(F)$ since the curve $(\varphi_G)_q$ is confined to F. Therefore, $p \in \tilde{U}'_H = \varphi_H(C_G(H) \times [0, \tilde{\varepsilon}))$ which is an open neighborhood of p. Hence, $p_n \in C(G) \cap \tilde{U}'_H = \tilde{U}_H$ for an n big enough which is a contradiction and $C(F,\eta) \cap \bigcup C(\mathcal{G}) \subset \tilde{U}$ for η small.

Now we are going to glue X and $\tilde{X}$ into a single vector field in the following way: Let $\chi : P \to [0,1]$ be a bump function supported in $\bigcup C(\mathcal{G})$ which is 1 on $\overline{\bigcup C(\mathcal{G}, \varepsilon')}$ for some $0 < \varepsilon' < \varepsilon$. Its existence is guaranteed by Lemma 19. We set $U_F = C(F, \eta)$ and define a vector field $X_F : U_F \to TU_F$ by the

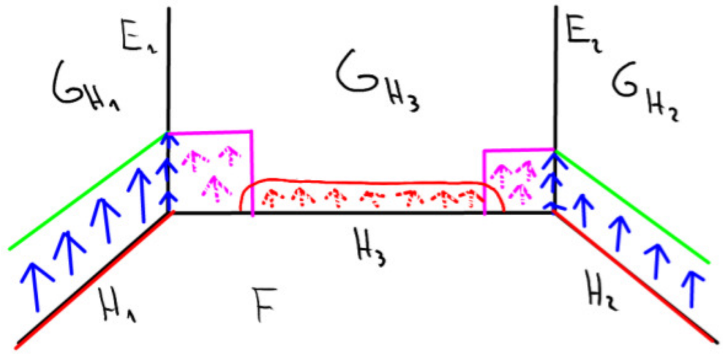

Figure 9: The situation $\mathcal{H} = \{H_1, H_2\}$ but $\mathcal{G} = \{G_{H_1}, G_{H_2}, G_{H_3}\}$. The vector fields X_{H_1} and X_{H_2} define vector fields along E_1 and E_2 in G_{H_3}. These are then used as vector fields associated to $\mathcal{H}' = \{E_1 \cap H_1, E_2 \cap H_2\}$ and are extended to a vector field on a neighborhood of H_3 in G_{H_3} using the inductive assumption. It is indicated by the dotted lines.

formula

$$X_F(p) = \chi(p)\tilde{X}(p) + (1 - \chi(p))\, X(p)$$

for $p \in U_F$. It is well defined and smooth because $U_F \cap \bigcup C(\mathcal{G}) \subset \tilde{U}$. It is normal to F since it is a sum of two vector fields normal to F. It equals $\tilde{X}$ on $\bigcup C(\mathcal{G}, \varepsilon') \cap U_F$ which is an open neighborhood of $\bigcup \mathcal{G} \cap U_F$. Therefore, X_F commutes with X_G restricted to $C(G, \varepsilon')$ for every $G \in \mathcal{G}$ as $\tilde{X}$ does; hence, $\{X_G\big|_{C(G,\varepsilon')} : G \in \mathcal{G}\} \cup \{X_F\}$ is a system of commuting normal vector fields. If X_H is restricted to $U_H \cap U_F$ then $X_F\big|_{G_H} = \tilde{X}\Big|_{G_H} = X_H$ on the common domain. Therefore, X_F is a vector field satisfying (i) and (ii) after shrinking of domains. This we have been looking for.

Now assume $G \in \mathcal{G}$ and $H \in \mathcal{F}^1(F) \cap \mathcal{F}^1(G)$ such that $H \notin \mathcal{H}$; in other words, we investigate the case $\mathcal{G} \neq \emptyset$ and $\mathcal{H} \subsetneq \mathcal{G}_F$. We restrict the systems of vector fields we work with from P to G as in Remark 26. These clearly satisfy all assumptions of the lemma on G which has a lower dimension than

P. Therefore, we may use the inductive step to construct a vector field X_H normal to H in G (See Figure 9). This vector field commutes with restrictions $\{X_{G'}\big|_{G'}, G' \in \mathcal{G}\}$ and equals $X_{H'}$ for $H' \in \mathcal{H}$ on overlap of domains by construction. Therefore, the systems of vector fields $\{X_{G'} : G \in \mathcal{G}\}$ and $\{X_H :\in \mathcal{H}\} \cup \{X_H\}$ satisfy all assumptions of the lemma on P. Repeated usage of this argument reduces the problem to the case $\mathcal{H} = \mathcal{G}_F$. $\square$

THEOREM 17 (Collar Neighborhood Theorem). *Let P be a compact manifold with embedded faces. Then there is a system of commuting normal vector fields $\{X_F : F \in \mathcal{F}^1(P)\}$ which generates a neat system of compatible collar neighborhoods $C(\mathcal{F}^1(P), 1)$.*

Proof. We start with $\mathcal{G} = \emptyset$ and use lemma 35 with $\mathcal{H} = \emptyset$ repeatedly to enlarge $\mathcal{G}$ with a new 1-face each time. We get $\mathcal{G} = \mathcal{F}^1(P)$ eventually. Therefore, there is a system of commuting normal vector fields $\{X_F : F \in \mathcal{F}^1(P)\}$. Lemma 34 asserts there is a neat system of collar neighborhoods $C(\mathcal{F}^1(P), \varepsilon)$ generated by $\{X_F : F \in \mathcal{F}^1(P)\}$. Now it is straightforward to compose φ_F with scaling $(p, t) \mapsto \left(p, \frac{1}{\varepsilon}\right)$ where $(p, t) \in F \times [0, \varepsilon)$ in order to get $C(\mathcal{F}^1(P), 1)$. The generating vector fields are recovered as $X_F(\varphi_F(p, t)) = (\mathrm{d}\varphi_F^t)_p(1)$. $\square$

2.11 Decompositions

In this section we will concern ourselves with decompositions. The idea of a decomposition comes from [1] and it is a geometric alternative to a simplicial complex. We may decompose a manifold with corners into manifolds with embedded faces as follows.

Definition 36 (Decomposition). *Let P be a manifold with corners. A* ***decomposition*** *of P is a locally finite close cover of P which we denote by $\mathcal{K}(P)$ (shortly $\mathcal{K}$) such that*

(a) Every $K \in \mathcal{K}$ is a connected manifold with embedded faces which is embedded in P and $\dim(K) = \dim(P)$*.*

(b) For $K, K' \in \mathcal{K}$ holds either $K \cap K' = \emptyset$ or $K \cap K'$ is a k-face of both K and K' for some $k \in \mathbb{N}_0$

(c) If $K \in \mathcal{K}$ and $F \in \mathcal{F}^k(P)$ for some $k \in \mathbb{N}_0$ then $K \cap F = \emptyset$ or $K \cap F$ is an l-face of K for some $l \in \mathbb{N}_0$.

If P is oriented and every $K \in \mathcal{K}$ is oriented such that $K \hookrightarrow P$ is orientation preserving then we say $\mathcal{K}$ is ***oriented consistently with*** *P.*

Remark 29 (Properties of Decompositions). (i) $\mathrm{Int}(K) \subset \mathrm{Int}(P)$ is an open submanifold and $d_K(p) \geq d_P(p)$ for all $p \in K$, $K \in \mathcal{K}$: Both properties come from Lemma 9.

(ii) If P is oriented and $\mathcal{K}$ is its arbitrary decomposition then every $K \in \mathcal{K}$ is orientable and may be oriented so that $K \hookrightarrow P$ is an orientation preserving immersion. Therefore, $\mathcal{K}$ can be always made to be consistently oriented: This follows from Lemma 24.

(iii) $\mathcal{K}$ is countable and if P is compact then $\mathcal{K}$ is finite : $\{\mathrm{Int}(K) : K \in \mathcal{K}\}$ is a set of non-empty disjoint open subspaces of P and we may pick an element of a countable basis in $\mathrm{Int}(K)$ for every $K \in \mathcal{K}$ by Lemma 4. Therefore, $\mathcal{K}$ is countable. If P is compact we may pick an open subspace $B_K \subset P$ such that $\overline{B_K} \subset \mathrm{Int}(K)$. The set $\{P \backslash \overline{\bigcup_{K \in \mathcal{K}} B_K}\} \cup \{\mathrm{Int}(K) : K \in \mathcal{K}\}$ is then an open cover of P since $\overline{\bigcup_{K \in \mathcal{K}} B_K} = \bigcup_{K \in \mathcal{K}} \overline{B}_K$ due to local finiteness of $\mathcal{K}$. Compactness of P implies this cover is finite; hence, $\mathcal{K}$ is finite.

Lemma 36 (Intersection of a Stratum and a Face in a Decomposition). *Let P be a manifold with corners and $\mathcal{K}$ a decomposition of P. Let $K \in \mathcal{K}$ and $S \in \mathcal{S}^l(K)$ for some $l \in \mathbb{N}_0$. Let $F \in \mathcal{F}^k(P)$ or $F \in \mathcal{F}^k(K')$ for some $K' \in \mathcal{K}$ and $k \in \mathbb{N}_0$. It holds: If $S \cap F \neq \emptyset$ then $S \subset F$ and $k \leq l$.*

Proof. Firstly, let $F \in \mathcal{F}^k(P)$. By axiom (iii) of Definition 36 there is a $G \in \mathcal{F}^m(K)$ for some m such that $K \cap F = G$. Then $\emptyset \neq S \cap F \subset K \cap F = G$ by the assumption and Lemma 6 implies $S \subset G$; in particular, $S \subset F$. Because $d_K(p) \geq d_P(p)$ for any $p \in K$ we have $k \leq l$.

Secondly, let $F \in \mathcal{F}^k(K')$ for some $K' \in \mathcal{K}$. Axiom (ii) of the definition implies there is a common m-face L of both K and K' such that $K \cap K' = L$. However, $\emptyset \neq S \cap F \subset K \cap K' = L$ by the assumption and it follows $S \subset L$ by Lemma 6 applied to K. Because K is a manifold with embedded faces, L is its m-face and $\overline{S}$ its l-face, it follows by theorem 7 that $\overline{S}$ is an $(l-m)$-face of L . The structure of a manifold with embedded faces on L is unique regardless whether it is considered as a face of K or K' because it embeds in P and lemma 10 applies. Therefore, $\overline{S}$ is an l-face and S is an l-stratum of K' as well. Because $\emptyset \neq S \cap F$ Lemma 6 applied to K' implies $S \subset F$ and $k \leq l$. The lemma is hereby proven. □

Lemma 37 (1-Faces of a Decomposition). *Let P be a manifold with embedded faces and let $\mathcal{K}$ be a decomposition of P. Let $K \in \mathcal{K}$. Then for every $H \in \mathcal{F}^1(K)$ precisely one of the following situations occurs:*

(a) (Boundary Face) It holds $\mathrm{Int}(H) \cap \partial P \neq \emptyset$ *and there exists a unique* $F \in \mathcal{F}^1(P)$ *such that* $H \subset F$.

(b) (Interior Face) It holds $\mathrm{Int}(H) \subset \mathrm{Int}(P)$ *and there exists a unique* $K' \in \mathcal{K}$, $K' \neq K$ *such that* $K \cap K' = H$ *is a common* 1*-face of both.*

In addition, suppose that P *is oriented and* $\mathcal{K}$ *is oriented consistently. Then the boundary orientation on* H *from* K *agrees with the boundary orientation of* F *from* P *when* H *is a boundary face and disagrees with the boundary orientation of* H *induced from* K' *when* H *is an interior face.*

Proof. Let $K \in \mathcal{K}$ and $H \in \mathcal{F}^1(K)$. If $\mathrm{Int}(H) \cap \partial P \neq \emptyset$ let $p \in \mathrm{Int}(H) \cap \partial P$. Because $d_P(p) = 1$ by (i) of Remark 29 there is a unique $S \in \mathcal{S}^1(P)$ such that $p \in S$. Lemma 36 asserts $\mathrm{Int}(H) \subset \overline{S}$; hence, $H \subset \overline{S}$ and the case (a) holds. Moreover, $T_p^+ K \subset T_p^+ P$ by Lemma 8 and we see inward pointing vector to H at p in K is an inward pointing vector to H at p in P. Therefore, the boundary orientations agree.

Suppose now $\mathrm{Int}(H) \subset \mathrm{Int}(P)$ and take a $p \in \mathrm{Int}(H)$. There has to be a $K' \in \mathcal{K}$, $K' \neq K$ such that $p \in K'$. If there was no such K' then, since the union of all $K' \neq K$ is closed, there would be an open neighborhood $U \subset \mathrm{Int}(P)$ of p such that $U \subset K$. This is a contradiction since p would be a boundary point of U considered as an open submanifold of K but U is an open submanifold of $\mathrm{Int}(P)$ without boundary. Having $p \in K'$, (iii) of Definition 36 asserts that $K' \cap K$ is an m-face of both K and K'. Nevertheless, there is only one possible face of K containing p since $p \in S^1(K)$. This face is H and we have $K \cap K' = H$. By the same reasoning as in the previous paragraph we see that $H \in \mathcal{F}^1(K')$. Existence of K' is hereby proven. Uniqueness of K' follows as we may easily find an open neighborhood $U \subset P$ of p such that $U = (U \cap \mathrm{Int}(K)) \cup (U \cap \mathrm{Int}(K')) \cup (U \cap \mathrm{Int}(H))$ using regular neighborhoods. Now if p was contained in an other K'' then U has to contain points from $\text{Int}(K'')$. This is a contradiction since such points do not fit anywhere in U. Therefore, the case (b) holds. Considering the orientation, it holds $T_p^+K' \cap T_p^+K = T_pH$ and $T_p^+K \cup T_p^+K' = T_pP$ since curves in P starting at p go either in $\text{Int}(K)$, or in $\text{Int}(K')$ or stay in H locally. From this we get $T_p^+K = -T_p^+K'$. The fact that the induced orientations on H disagree is now easy to deduce. $\square$

THEOREM 18 (Induced Decomposition of the Boundary). *Let P be a manifold with embedded faces and $\mathcal{K}(P)$ a decomposition of P. Let $F \in \mathcal{F}^k(P)$ for some $k \in \mathbb{N}_0$. Then*

$$\mathcal{K}(F) = \{G \in \mathcal{F}^k(K) : G \subset F, K \in \mathcal{K}(P)\}$$

is a decomposition of F.

Proof. We prove the theorem by induction on dimension of P and for fixed dimension by the induction on k :

If $\dim(P) = 0$ there are no faces and there is nothing to prove. Assume $\dim(P) > 0$ and that the lemma holds for all smaller dimensions. If $k = 0$ then the lemma trivially holds as 0-faces are the manifolds themselves. Let $k > 0$ and assume the lemma holds for all smaller k :

Let $F \in \mathcal{F}^k(P)$. There is a $G \in \mathcal{F}^{k-1}(P)$ such that $F \in \mathcal{F}^1(G)$ by Theorem 7. By the inductive assumption on k the set

$$\mathcal{K}(G) = \{H \in \mathcal{F}^{k-1}(K) : H \subset G, K \in \mathcal{K}(P)\}$$

is a decomposition of G. By the inductive assumption on dimension the set

$$\mathcal{K}'(F) = \{E \in \mathcal{F}^1(H) : E \subset F, H \in \mathcal{K}(G)\}$$

is a decomposition of F. Clearly, $\mathcal{K}'(F) \subset \mathcal{K}(F)$ by theorem 7 again. In order to prove the opposite inclusion let $K \in \mathcal{K}(P)$ and $E \in \mathcal{F}^k(K)$ be such that $E \subset F$. Because $\mathcal{K}'(F)$ is a decomposition of F there is an $E' \in \mathcal{K}'(F)$ such that $E' \cap \mathrm{Int}(E) \neq \emptyset$. Let $K' \in \mathcal{K}(P)$ be such that $E' \in \mathcal{F}^k(K')$ by the definition of $\mathcal{K}'(F)$ and $\mathcal{K}(G)$. Lemma 36 implies $E \subset E'$. Because $K \cap K' = L$ is an l-face of both K and K' by the definition of decomposition and $E, E' \subset L$ are both $(k-l)$-faces of L we get $E = E'$. Therefore, $\mathcal{K}(F) = \mathcal{K}'(F)$ and the lemma is proven. $\square$

Definition 37 (Geometric Triangulation / Cubification). *Let P be a manifold with embedded faces. A* ***geometric triangulation / cubification*** *of P is a decomposition $\mathcal{T}$ of P such that every $K \in \mathcal{T}$ is diffeomorphic to Δ^n / $[0,1]^n$ where $n = \dim(P)$.*

Remark 30 (Simplicial Complex). Suppose that if K, $K' \in \mathcal{K}$, $\psi : K \to \Delta^n$ and $\varphi : K' \to \Delta^n$ are the identifying diffeomorphisms, and $K \cap K' = L$ is an l-face of both then $\psi \circ \varphi^{-1} : \Delta^{n-l} \to \Delta^{n-l}$ is a linear isomorphism. Then it is possible that elements of $\mathcal{K}$ glue to a simplicial complex.

Conjecture 1 (Extension of Triangulation / Cubification from Boundary). *Let P be a compact connected manifold with embedded faces. Let $\mathcal{G} \subset \mathcal{F}^1(P)$ be such that there is a geometric triangulation / cubification $\mathcal{T}_G(G)$ of G for every $G \in \mathcal{G}$. In addition, suppose that for every $G, G' \in \mathcal{G}$ and for every common 1-face H the following compatibility condition hold*

$$\mathcal{T}_G(H) = \mathcal{T}_{G'}(H).$$

Then there is a geometric triangulation / cubification $\mathcal{T}(P)$ of P such that for every $G \in \mathcal{G}$ holds

$$\mathcal{T}(G) = \mathcal{T}_G(G).$$

3 Geometric Homology

In this section we will define geometric simplexes as maps from compact manifolds with embedded faces into a topological space X up to a diffeomorphism. Subsequently, we will define geometric chains and the boundary operator. We will prove the boundary operator squares to zero using results from the previous chapter. We will end up this section with a definition of the geometric homology theory $\mathcal{HP}(X)$. The ideas are based on [1] with only slight technical differences.

Other notions of generalized homology theories may be found in the literature. For example, a homology and cohomology theory of Whitney stratified objects is constructed in [13]. In spite of the fact that manifolds with corners are naturally stratified objects, we prefer working with them as with smooth manifolds because of the advantage of having a smooth structure on the whole space. Another article containing thoughts about homology using manifolds with corners is [14], for instance.

3.1 Geometric Chains and Homology

X will be a topological space and R a ring throughout this section.

DEFINITION 38 (Geometric Simplex). *Consider ordered pairs (P, f) where P is an oriented compact connected manifold with embedded faces and $f : P \to X$ a continuous map. Two pairs (P, f) and (Q, g) are called* ***equivalent*** *if there is an orientation preserving diffeomorphism $\psi : Q \to P$ such that $g \circ \psi = f$. An equivalence class $[P, f]$ of such pairs is called a* ***geometric simplex in*** *X.*

REMARK 31 (Set of Geometric Simplexes). The class of manifolds with corners is a proper class since every set S can be considered as a different manifold with corners diffeomorphic to the one-point manifold $\{*\}$. However, every compact manifold with corners of a specific dimension is diffeomorphic to an embedded submanifold of some Euclidean space $\mathbb{R}^N$ as we have seen in Theorem 9. This subspace has a unique structure of a manifold with corners by Lemma 10. It follows, there is a set of compact manifolds with corners such that every compact manifold with corners is diffeomorphic to an element of the set. Consequently, the same conclusion is true for pairs (P, f); namely, there is a set of non-equivalent representants (P, f) which determine all equivalence classes $[P, f]$. We identify geometric simplexes with these representants and talk about the *set of geometric simplexes.*

DEFINITION 39 (Boundary of a Geometric Simplex). *Denote by $\mathcal{P}^0(X; R)$ (shortly $\mathcal{P}^0$) the free module over R with basis the set of geometric simplexes in X. For a geometric simplex $[P, f]$ we define its* ***boundary*** *$\partial[P, f] \in \mathcal{P}^0(X; R)$ by the formula*

$$\partial[P, f] = \sum_{i=1}^{k} [\partial_i P, \partial_i f]$$

where $\mathcal{F}^1(P) = \{\partial_i P : i = 1, \ldots, k\}$ with the boundary orientation.

REMARK 32 (Properties of $\mathcal{P}^0$ and ∂). (i) $\partial[P, f]$ is well defined: An orientation preserving diffeomorphism $\psi : P \to Q$ of manifolds with embedded faces restricts to diffeomorphisms $\psi_i : \partial_i P \to \partial_i Q$ for $i = 1, \ldots, k$ preserving boundary orientations. As a result, if $[P, f] = [Q, g]$ then $\partial[P, f] = \partial[Q, g]$.

(ii) The module $\mathcal{P}^0$ is naturally graded by the dimension $d \geq 0$. We write

$$\mathcal{P}^0 = \bigoplus_{d=0}^{\infty} \mathcal{P}_d^0$$

where the module $\mathcal{P}_d^0$ is generated by geometric simplexes of dimension d. Note that $\mathcal{P}_0^0$ is generated by $\{[\{*\}, x] : x \in X\}$ and that if one wishes $\emptyset$ may be considered as an oriented manifold in every dimension; hence, $[\emptyset, \emptyset] \in \mathcal{P}_d^0$ for every d. The boundary operator extends linearly from basis to a degree -1 endomorphism

$$\partial : \mathcal{P}_0^0 \to \mathcal{P}_0^0.$$

(iii) Note that we may allow disconnected manifolds in the definition of a geometric simplex. Then we need to factor out sums of connected components in $\mathcal{P}^0$. If we do this, we may write $\partial[P, f] = [\tilde{\partial} P, f]$ where $\tilde{\partial}$ is the abstract boundary from Definition 7.

DEFINITION 40 (Degenerate Simplex). *A geometric simplex* $[P, f]$ *of positive dimension is called* ***degenerate*** *if there exists a geometric simplex* $[P', f']$ *with* $\dim(P') = \dim(P) - 1$ *and a smooth map* $\psi : P \to P'$ *such that*

$$f = f' \circ \psi \qquad \text{and} \qquad d_{P'}(\psi(p)) = d_P(p)$$

for every $p \in P$.

REMARK 33 (Boundary of a Degenerate Simplex). If $\dim(P) \geq 2$ and $F \in \mathcal{F}^1(P)$ then $[F, f\big|_F]$ is degenerate as well since the degeneracy map ψ restricts to a smooth map $\psi : F \to G$ where $G \in \mathcal{F}^1(Q)$. Indeed, $\psi(F) \subset G$ from depth-preservation and continuity, and the map is smooth since faces are embedded and (i) of Remark 8 applies.

DEFINITION 41 (Geometric Chains). *We define the module $\mathcal{P}(X;R)$ (shortly $\mathcal{P}$) as the module over R generated by all geometric simplexes in X with the following relations imposed for every $[P,f]$:*

(i) $[-P,f] = -[P,f]$.

(ii) $[P,f] = 0$ *whenever* $[P,f]$ *is degenerate.*

(iii) $[P,f] = \sum_{i=1}^{l} [P_i, f_i]$ *where* $\{P_1,\ldots,P_l\}$ *is a consistently oriented decomposition of* P.

*The elements of $\mathcal{P}(X;R)$ are called **geometric chains in** X.*

REMARK 34 (Properties of Geometric Chains). (i) Note that $\mathcal{P}$ is constructed as a factor module of the free module $\mathcal{P}^0$ by a submodule generated by relations (i) to (iii) above.

(ii) In contrast to the singular chain complex the module $\mathcal{P}$ is not free: We take $X = [0,1]$, $P = [-1,1]$, and $f : P \to X$ and $\psi : P \to P$ defined by formulas $f(x) = |x|$ and $\psi(x) = -x$ for every $x \in P$. Then ψ is an orientation preserving diffeomorphism of P and $-P$ such that $f \circ \psi = f$. From the definition of a geometric simplex we see $[P,f] = [-P,f]$; thus, after factoring out relation (i) we get in $\mathcal{P}$

$$2[P,f] = 0.$$

THEOREM 19 (The Boundary Squares to Zero). *The boundary map ∂ restricts to an endomorphism $\partial : \mathcal{P} \to \mathcal{P}$ and it holds $\partial \circ \partial = 0$. Therefore, the pair $(\mathcal{P}, \partial)$ is a chain complex.*

Proof. In order to prove that ∂ is well-defined on $\mathcal{P}$ it suffices to check that $\partial : \mathcal{P}^0 \to \mathcal{P}^0$ maps relations to relations: For a geometric simplex $[P, f]$ we have by (iii) of Remark 17

$$\partial[-P, f] = -\partial[P, f]$$

If $[P, f]$ is degenerate and $\dim(P) \geq 2$ then $[\partial_i P, \partial_i f]$ is degenerate by Remark 33 for every $i = 1, \ldots, k$. If $\dim(P) = 1$ then compact connected P is either $\mathbb{S}^1$ or $[0, 1]$ by the classification theorem for 1-manifolds (see [15, p. 55-57] for example). If $P = \mathbb{S}^1$ then there is no face and

$$\partial[\mathbb{S}^1, f] = 0.$$

Let $P = [0, 1]$. The degeneracy means $f : P \to X$ is equal to a constant $x \in X$. P can be positively oriented either to the left or to the right. In both cases $\varepsilon(0)$ and $\varepsilon(1)$ have opposite sign by Remark 16. Therefore,

$$\partial[P, x] = [\{*\}, x] - [\{*\}, x] = 0$$

Now let $[P, f]$ be arbitrary and let $\mathcal{K}$ be a consistently oriented decomposition of P. Then we may calculate

$$\begin{aligned} \partial\left(\sum_{K\in\mathcal{K}}[K, f\big|_K]\right) &= \sum_{K\in\mathcal{K}}\sum_{H\in\mathcal{F}^1(K)}[H, f\big|_H] = \qquad (27)\\ &= \sum_{F\in\mathcal{F}^1(P)}\sum_{\substack{H\in\mathcal{F}^1(K), H\subset F\\ K\in\mathcal{K}}}[H, f\big|_H] + \sum_{\substack{H\in\mathcal{F}^1(K)\cap\mathcal{F}^1(K')\\ K,K'\in\mathcal{K}, K\neq K'}}[H, f\big|_H] + [-H, f\big|_H] \end{aligned}$$

where the first equality is the definition and the second is precisely classification of 1-faces of a decomposition from Lemma 37. The first sum is precisely the sum of decompositions of 1-faces of P according to Theorem 18. Therefore, we see that ∂ maps relations into relations and restricts to an endomorphism of $\mathcal{P}$. In particular, for an arbitrary decomposition $\mathcal{K}$ of P holds

$$\partial[P, f] = \partial\left(\sum_{K \in \mathcal{K}} [K, f\big|_K]\right)$$

in $\mathcal{P}$ since the last term in Equation 27 becomes zero in $\mathcal{P}$.

Now it remains to show that $\partial \circ \partial = 0$ on $\mathcal{P}$. It suffices to check it for an arbitrary generator $[P, f]$. We may calculate

$$\begin{aligned}(\partial \circ \partial)[P, f] &= \sum_{F \in \mathcal{F}^1(P)} \sum_{H \in \mathcal{F}^1(F)} [H, f\big|_H] = \sum_{\substack{H \in \mathcal{F}^2(P), H \subset F \\ F \in \mathcal{F}^1(P)}} [H, f\big|_H] = \\ &= \sum_{H \in \mathcal{F}^2(P)} [H, f\big|_H] + [-H, f\big|_H] \end{aligned} \tag{28}$$

where the first equality is the definition of ∂ applied two times, the second equality is due to Theorem 7, and the third by Lemma 23. The right hand side of Equation 28 is a linear combination of relations and becomes zeros in $\mathcal{P}(X; R)$. The theorem is hereby proven. $\square$

Definition 42 (Geometric Homology). *We define the* ***geometric homology of*** X ***with coefficients in*** R *as the homology theory* $\mathcal{H}P(X; R)$ *associated to the chain complex* $(\mathcal{P}(X; R), \partial)$.

Conjecture 2 (Equivalence of Geometric and Singular Homology). *Let* X *be a CW complex and* $(C(X; R), \partial)$ *the singular chain complex of* X*. Then the natural chain map*

$$\begin{array}{rrcl}\psi: & C(X; R) & \to & \mathcal{P}(X; R) \\ & (f : \Delta^n \to X) & \mapsto & [\Delta^n, f]\end{array}$$

induces an isomorphism

$$\mathcal{H}\psi : \mathcal{H}(X; R) \to \mathcal{H}\mathcal{P}(X; R).$$

4 Conclusion and Further Work

As it was already mentioned, the original goal of this thesis was to prove equivalence of geometric and singular homology and apply geometric homology to the string topology. I had thought about these problems at the beginning of my work before I realized that without sufficient knowledge of manifolds with corners I would not be able to write a rigorous theory. At that point I found out there was lack of references about manifolds with corners and the necessity to examine them closer arose.

The plan of further work is the following: First of all, I will use the collar neighborhood Theorem 17 and the theory of triangulations of manifolds without boundary from [16] to prove the triangulation conjecture 1. Note that it might be easier to consider cubifications instead of triangulations because of the shape of collar neighborhoods. Secondly, I will apply this result to prove that geometric homology of compact manifolds with embedded faces is isomorhpic to the geometric homology generated by geometric simplexes of the form $[\Delta^q, f]$ where Δ^q is a standard q-simplex. Lastly, I will show that the latter homology theory satisfies Eilenberg-Steenrod Axioms; hence, it is naturally isomorphic to the singular homology on CW pairs. All the steps above are outlined in [1]. At this point the geometric homology theory will be ready to apply to string topology.

5 Appendix - Analysis on $\mathbb{R}^q_+$

Definition 43 (Smooth Map). *Let $A \subset \mathbb{R}^q$, $B \subset \mathbb{R}^p$ be arbitrary subspaces and $f : A \to B$ a map. Then f is called **smooth** if the following holds: For every $x \in A$ there is an open neighborhood $U \subset \mathbb{R}^q$ of x and a map $g : U \to \mathbb{R}^p$ such that*

(a) $g : U \to \mathbb{R}^p$ has partial derivatives of all orders everywhere on U.

(b) $g \equiv f$ on the common domain.

*The map $g : U \to \mathbb{R}^p$ is called a **smooth extension of** f **at** x.*

Remark 35 (Properties of Smooth Maps). (i) A smooth map is continuous.

(ii) Smoothness is a local property of a map $f : A \to B$.

(iii) Composition of smooth maps $A \xrightarrow{f} B \xrightarrow{g} C$ is smooth.

Definition 44 (Differential). *Let $A \subset \mathbb{R}^q$, $B \subset \mathbb{R}^p$ be arbitrary subspaces, $x \in A$ and let $f : A \to B$ be a smooth map. Then f is called **differentiable at** x if the following condition holds: Whenever $g : U \to \mathbb{R}^p$ and $g' : U' \to \mathbb{R}^p$ are two smooth extensions of f at x then the differentials $\mathrm{d}g_x, \mathrm{d}g'_x : \mathbb{R}^q \to \mathbb{R}^p$ agree. In this case, we denote the linear map $\mathrm{d}f_x = \mathrm{d}g_x = \mathrm{d}g'_x : \mathbb{R}^q \to \mathbb{R}^p$ and call it the **differential of** f **at** x. When $q = 1$ we denote the differential of $f : A \to B$ at $t \in A$ by $\frac{\mathrm{d}}{\mathrm{d}t} f(t)$.*

Remark 36 (Chain Rule and Differentiability). (i) Let $f : A \to B$ and $g : B \to C$ be smooth maps and $x \in A$ such that f is differentiable at x and g is differentiable at $f(x)$. Then $g \circ f : A \to C$ is differentiable at x and it holds $\mathrm{d}(g \circ f)_x = \mathrm{d}g_{f(x)} \circ \mathrm{d}f_x$.

(ii) Let $f : A \to B$ be a smooth map. If there is an open subspace $W \subset \mathbb{R}^q$ such that $W \subset A \subset \overline{W}$ then f is differentiable everywhere : Clearly, f is differentiable at all points $x \in W$. Let $x \in A$ be a boundary point of W and let $(x_n)_{n\in\mathbb{N}} \subset W$ be a sequence such that $x_n \mapsto x$ for $n \mapsto \infty$. Let $g : U \to \mathbb{R}^p$ be a smooth extension of f at x and let $n_0 \in \mathbb{N}$ be such that $x_n \in U$ for $n \geq n_0$. If $n \geq n_0$ then $\mathrm{d}f_{x_n} = \mathrm{d}g_{x_n}$ since the maps agree on an open neighborhood of x_n in $\mathbb{R}^q$. Because g is smooth, its partial derivatives at x can be calculated as limits of partial derivatives at x_n which are determined by f. Therefore, $\mathrm{d}g_x$ is determined purely by f by uniqueness of limits and $\mathrm{d}g_x = \mathrm{d}\tilde{g}_x$ for any extension $\tilde{g}$.

Definition 45 (Euclidean Corner). *The q-**corner** $\mathbb{R}^q_+$ is defined as*

$$\mathbb{R}^q_+ = \{x \in \mathbb{R}^q : x \geq 0\}$$

For $x \in \mathbb{R}^q_+$ we denote

$$I_x = \{i \in \{1, \dots, q\} : x^i = 0\}$$

*and define the **depth of** x as the number*

$$d(x) = |I_x| = \text{"the number of zero coordinates of } x\text{"}$$

*For an open subspace $U^+ \subset \mathbb{R}^q_+$ and $I \subset \{1, \dots, q\}$ we define the I-**strata of** U^+ as*

$$\partial^0_I U^+ = \{x \in U^+ : x^I = 0,\, x^{I^c} > 0\}$$

*and the I-**face of** U^+ as*

$$\partial_I U^+ = \{x \in U^+ : x^I = 0\}.$$

*Here $x^J > 0$ means $x^j > 0$ for all $j \in J$ and similarly for the relations. Cardinality $|I|$ is called the **co-dimension** of the I-face / I-strata. We also define the **boundary** and the **interior** of U^+ as subspaces of U^+ given by formulas*

$$\partial U^+ = \bigcup_{i=1}^{q} \partial_i U^+ \qquad \text{and} \qquad \mathrm{Int}(U^+) = \partial^0_\emptyset U^+.$$

Remark 37 (Properties of Open Subspaces). Note that $\mathrm{Int}(U^+) \subset U^+ \subset \overline{\mathrm{Int}(U^+)}$. Therefore, $U^+ = \emptyset$ if and only if $\mathrm{Int}(U^+) = \emptyset$, U^+ is connected if and only if $\mathrm{Int}(U^+)$ is connected, and any smooth function $f : U^+ \to V^+$ is differentiable everywhere by (ii) of Remark 36.

Definition 46 (Regular Neighborhood of a Point). *Let $x \in \mathbb{R}^q_+$ and let $U^+ \subset \mathbb{R}^q_+$ be an open neighborhood of x. Then U^+ is called a* ***regular neighborhood of*** *x if it holds $\partial_i U^+ = \emptyset$ for every $i \notin I_x$.*

Remark 38 (Existence of Regular Neighborhoods). Clearly, every neighborhood U^+ of x in $\mathbb{R}^q_+$ contains a regular neighborhood of x: For $i \notin I_x$ we have $x^i > 0$ and we may intersect U^+ with $\{z \in \mathbb{R}^q_+ : z^i > \frac{x^i}{2}, i \notin I_x\}$.

Lemma 38 (Tangent Vectors to Curves). *Let $U^+ \subset \mathbb{R}^q_+$ be an open subspace and $x \in U^+$. We denote $T_x U^+ = \mathbb{R}^q$ and define*

$$C_x U^+ = \{\gamma'(0) \in T_x U^+ : \gamma : (-\varepsilon, \varepsilon) \to U^+ \text{ a smooth curve with } \gamma(0) = x\}.$$

$$T^+_p U^+ = \{\gamma'(0) \in T_x U^+ : \gamma : [0, \varepsilon) \to U^+ \text{ a smooth curve with } \gamma(0) = x\}.$$

Then the following holds:

(a) $C_x U^+ = \{v \in T_x U^+ : v^{I_x} = 0\}$, $T^+_x U^+ = \{v \in T_x U^+ : v^{I_x} \geq 0\}$ and $C_x U^+$ inherits structure of a vector space with $\dim(C_x U^+) = q - d(x)$.

(b) If $f^+ : U^+ \to V^+$ is a smooth map of open subspaces then

$$\mathrm{d}f^+_x(C_x U^+) \subset C_{f^+(x)} V^+ \qquad \text{and} \qquad \mathrm{d}f^+_x(T^+_x U^+) \subset T^+_{f^+(x)} V^+.$$

Proof. Let $\gamma : (-\varepsilon, \varepsilon) \to U_+$ be a smooth curve as in the definition of $C_x U^+$. Then $\left.\frac{\mathrm{d}\gamma^i(t)}{\mathrm{d}t}\right|_{t=0} = 0$ for every $i \in I_x$: If $\left.\frac{\mathrm{d}\gamma^i(t)}{\mathrm{d}t}\right|_{t=0} \neq 0$ then the definition of derivative implies there is a $t_0 \in (-\varepsilon, \varepsilon)$ with $\gamma^i(t_0) < 0$. However, this would contradict the assumption that $\gamma(t) \geq 0$ for all $t \in (-\varepsilon, \varepsilon)$. On the other hand, let $v \in \mathbb{R}^q$ be a vector with $v^{I_x} = 0$. Since U^+ is open there is an open ball $B_\epsilon(x)$ in $\mathbb{R}^q$ centered at x of some radius $\varepsilon > 0$ such that $B_\varepsilon(x) \cap \mathbb{R}^q_+ \subset U^+$. By shrinking the ball if necessary we may assume that $\frac{1}{\varepsilon} > \|v\|$. Then the formula $\gamma(t) = x + tv$ for $t \in (-\varepsilon, \varepsilon)$ defines a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to U^+$ with $\gamma(0) = x$ and $\dot\gamma(0) = v$. Therefore, $C_x U^+ = \{v \in \mathbb{R}^q : v^{I_x} = 0\}$. Similarly, a smooth curve $\gamma : [0, \varepsilon) \to U^+$, $\gamma(0) = x$ satisfies $\frac{\mathrm{d}\gamma^i(t)}{\mathrm{d}t}\Big|_{t=0} \geq 0$ for all $i \in I_x$ otherwise there is a small positive t_0 with $\gamma^i(t_0) < 0$. Therefore, $T^+_x U^+ = \{v \in \mathbb{R}^q : v^{I_x} \geq 0\}$ and part (a) is proven hereby.

Consider now a smooth map $f^+ : U^+ \to V^+$ from (b). Let $v \in C_x U^+$ and let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^q_+$ be a smooth curve going through x with $\dot\gamma(0) = v$. Then $f^+ \circ \gamma : (-\varepsilon, \varepsilon) \to V^+$ is a smooth curve with $(f^+ \circ \gamma)'(0) = \mathrm{d}f^+_x(\dot\gamma(0)) = \mathrm{d}f^+_x(v)$ by the chain rule. Therefore, $v \in C_{f^+(x)} V^+$ and similarly for $T^+_x U^+$. This proves part (b) and the lemma is proven. $\square$

Lemma 39 (Depth-Rank Inequality). *Let $U^+ \subset \mathbb{R}^q_+$, $V^+ \subset \mathbb{R}^p_+$ be open subspaces and $f^+ : U^+ \to V^+$ a smooth map. For every $x \in U^+$ it holds*

$$p - \mathrm{rk}(\mathrm{d}f^+_x) \geq d(f^+(x)) - d(x).$$

Proof. Denote by R a direct complement of $\mathrm{Ker}(\mathrm{d}f^+_x)$ in $\mathbb{R}^q$. Then $\mathrm{d}f^+_x\big|_R : R \to \mathbb{R}^p$ is injective and $\dim(R) = \mathrm{rank}(\mathrm{d}f^+_x)$. Therefore, we have

$$\dim(R \cap C_x U^+) = \dim(\mathrm{d}f^+_x(R \cap C_x U^+)) \leq \dim(C_{f^+(x)} V^+)$$

since $\mathrm{d}f^+_x(C_x U^+) \subset C_{f^+(x)} V^+$ by Lemma (38). Using the Dimension of Sum and Intersection Formula we have

$$\dim(R \cap C_x U^+) + \dim(R + C_x U^+) = \dim(R) + \dim(C_x U^+).$$

Employing the estimates $\dim(R + C_x U^+) \leq q$ and $\dim(R \cap C_x U^+) \leq p - d(f^+(x))$ derived above we get the desired inequality. $\square$

Definition 47 (Diffeomorphism). *Let $U^+ \subset \mathbb{R}^q_+$, $V^+ \subset \mathbb{R}^p_+$ be open subspaces and $f : U^+ \to V^+$ a map. f is called a* ***diffeomorphism*** *if it is bijective and both f and f^{-1} are smooth.*

Lemma 40 (Invariance of Corner Points). *Let $U^+ \subset \mathbb{R}^q_+$, $V^+ \subset \mathbb{R}^p_+$ be open subspaces and let $f^+ : U^+ \to V^+$ be a diffeomorphism. Then $q = p$ and $d(f^+(x)) = d(x)$ for all $x \in U^+$.*

Proof. The chain rule applied to identities $f^+ \circ {f^+}^{-1} = {f^+}^{-1} \circ f^+ = \mathrm{id}$ implies that $\mathrm{d}f^+_x$ is an isomorphism with inverse $\mathrm{d}{f^+}^{-1}_{f(x)}$ for every $x \in U^+$. Therefore, $q = p$. We finish the proof by applying Lemma (39) to $f^+ : U^+ \to V^+$ and ${f^+}^{-1} : V^+ \to U^+$. □

Theorem 20 (Inverse Function Theorem for $\mathbb{R}^q_+$). *Let $U^+, V^+ \subset \mathbb{R}^q_+$ be open subspaces, $U^+ \neq \emptyset$, and let $f^+ : U^+ \to V^+$ be a smooth map. Let $x \in U^+$ be such that both of the following conditions are satisfied:*

(a) $\mathrm{d}f^+_x : \mathbb{R}^q \to \mathbb{R}^q$ *is an isomorphism*

(b) There is an open neighborhood $W^+ \subset U^+$ of x such that

$$f(\partial W^+) \subset \partial V^+$$

Then there are open neighborhoods $\tilde{U}^+ \subset U^+$ of x and $\tilde{V}^+ \subset V^+$ of $f(x)$ such that $f^+\big|_{\tilde{U}^+} : \tilde{U}^+ \to \tilde{V}^+$ is a diffeomorphism.

Proof. Smoothness and properties (a) and (b) remain valid whenever we shrink to a smaller open neighborhood of x in $\mathbb{R}^q_+$. Therefore, we may assume there are open subspaces U, $V \subset \mathbb{R}^q$ such that $U^+ = U \cap \mathbb{R}^q_+$, $V^+ = V \cap \mathbb{R}^q_+$, V^+ is connected, $f^+(\partial U^+) \subset \partial V^+$, and that there is a diffeomorphism $f : U \to V$ such that $f^+ = f\big|_{U^+}$. Existence of the diffeomorphism $f : U \to V$ follows from the standard inverse function theorem applied to an arbitrary smooth extension of f^+ at x. We see that the only fact which needs to be proven is that $f(U^+) = V^+$:

Pre-image of the connected space $\mathrm{Int}(V^+)$ under the homeomorphism $f : U \to V$ is connected and can not intersect ∂U^+ due to $f^+(\partial U^+) \subset \partial V^+$. As a consequence, it has to lie entirely either in $\mathrm{Int}(U^+) = \{x \in U : x > 0\}$ or in $U \backslash U^+ = \{x \in U : x < 0\}$ as these subspaces are clearly separated. We have $\mathrm{Int}(U^+) \neq \emptyset$ since $U^+ \neq \emptyset$ and $f(\mathrm{Int}(U^+)) \subset \mathrm{Int}(V^+)$ since $d(f^+(y)) \leq d(y)$ for all $y \in U^+$ by Lemma 9. Therefore, $f^{-1}(\mathrm{Int}(V^+)) \cap \mathrm{Int}(U^+) \neq \emptyset$; consequently, $f^{-1}(\mathrm{Int}(V^+)) \subset \mathrm{Int}(U^+)$. This together with $f(\mathrm{Int}(U^+)) \subset \mathrm{Int}(V^+)$ gives $f(\mathrm{Int}(U^+)) = \mathrm{Int}(V^+)$. The open subspace $U \backslash U^+ \subset U$ is mapped by f to an open subspace in V lying in $V \backslash \mathrm{Int}(V^+)$. Therefore, in can not contain any limit points of $\mathrm{Int}(V^+)$, in particular ∂V^+, and has to be wholly contained in $V \backslash V^+$ since $V^+ = \mathrm{Int}(U^+) \cup \partial V^+$. We derived so far $f(\mathrm{Int}(U^+)) = \mathrm{Int}(V^+)$ and $f(U \backslash U^+) \subset V \backslash V^+$. This together with $f(U) = V$ and the assumption $f(\partial U^+) \subset \partial V^+$ gives $f(U^+) = V^+$ and the proof is complete. $\square$

Lemma 41 (Topological Invariance). *Let $\emptyset \neq U^+ \subset \mathbb{R}^q_+$, $V^+ \subset \mathbb{R}^p_+$ be open subspaces and $f^+ : U^+ \to V^+$ a homeomorphism. Then $q = p$ and $f^+(\partial U^+) \subset \partial V^+$.*

Proof. The fact that $q = p$ follows from Brouwer's Invariance of Domain Theorem for $\mathbb{R}^q$ applied to the restriction of φ^+ to $\mathrm{Int}(U^+) \neq \emptyset$. The second fact follows from topological invariance of boundary of $\mathbb{H}^q \simeq \mathbb{R}^q_+$. $\square$

Theorem 21 (Local Form of a Submersion). *Let $U^+ \subset \mathbb{R}^q_+$, $V^+ \subset \mathbb{R}^p_+$ be open subspaces, $x \in U^+$, and let $f^+ : U^+ \to V^+$ be a smooth map. Suppose that $\mathrm{d}f^+_x : \mathbb{R}^q \to \mathbb{R}^p$ is surjective and that one of the following additional conditions holds:*

(a) $\mathrm{d}f_x^+\big|_{C_xU^+} : C_xU^+ \to \mathbb{R}^p$ *is surjective.*

(b) There is an open neighborhood $W^+ \subset U^+$ *of* x *such that*

$$f(\partial W^+) \subset \partial V^+.$$

(c) $\mathrm{d}f_x^+\big|_{C_xU^+} : C_xU^+ \to C_{f^+(x)}V^+$ *is surjective and there exist an open neighborhood* $W^+ \subset U^+$ *of* x *and indices* $j_k \in I_x$ *for* $k = 1, \dots, d(f^+(x))$ *such that* $f(\partial_{j_k} W^+) \subset \partial_{i_k} V^+$ *where* $I_{f^+(x)} = \{i_1, \dots, i_{d(f^+(x))}\}$.

Then there is an open neighborhood $\tilde{U}^+ \subset U^+$ *of* x*, an open subspace* $U'^+ \subset \mathbb{R}^q_+$*, and a diffeomorphism* $\psi^+ : \tilde{U}^+ \to U'^+$ *such that*

$$f^+ \circ \psi^{+-1} = \pi\big|_{U'^+} \tag{29}$$

where $\pi : \mathbb{R}^q \to \mathbb{R}^p$ *is the canonical projection to the first* p *coordinates.*

Proof. Smoothness and properties (a), (b), and (c) remain valid whenever we shrink to a smaller open neighborhood of x in $\mathbb{R}^q_+$. Therefore, we may assume there are open subspaces $U \subset \mathbb{R}^q$, $V \subset \mathbb{R}^p$ and a smooth map $f : U \to V$ such that $U^+ = U \cap \mathbb{R}^q_+$ is a regular neighborhood of x, $V^+ = V \cap \mathbb{R}^p_+$, and $f^+ = f\big|_{U^+}$.

An invertible map $\psi : \mathbb{R}^q \to \mathbb{R}^p$ which satisfies

$$\psi(z) = (f(z), z^K) \tag{30}$$

on a neighborhood of x where $K \subset \{1, \dots, q\}$, $|K| = q - p$ clearly fulfills Equation 29. Therefore, we only need to find a right K for which assumptions of the Inverse Function Theorem 20 are satisfied at x so that $\psi^+ = \psi\big|_{\mathbb{R}^q_+}$ becomes a diffeomorphism of an open neighborhood $\tilde{U}^+$ of x and an open subspace $U'^+ = \psi^+(\tilde{U}^+)$. More precisely, these conditions are satisfied if the vectors $\partial_i f(x)$, $i \notin K$ are linearly independent and if $z \in \partial U^+$ and

$f(z) \notin \partial V^+$ together imply $z^K \in \partial \mathbb{R}_+^{q-p}$. The last condition can be rephrased as: $z \in \partial U^+$ and $z^K \notin \partial \mathbb{R}_+^{q-p}$ together imply $f(z) \in \partial V^+$.

Consider $\mathrm{d}f_x$ expressed in terms of the Jacobian matrix

$$\mathrm{d}f_x = \begin{pmatrix} \partial_1 f^1(x) & \dots & \partial_q f^1(x) \\ \dots & \dots & \dots \\ \partial_1 f^p(x) & \dots & \partial_q f^p(x) \end{pmatrix}$$

Using Lemma 38 we see that $\mathrm{d}f_x(C_x U^+) = \{\partial_j f(x) : j \notin I_x\} \subset C_{f(x)} V^+$.

Condition (a) guarantees $\{\partial_j f(x) : j \notin I_x\} = \mathbb{R}^p$; thus, $|I_x| \leq q - p$ and we may choose $q - p$ indices $K \subset \{1, \dots, q\}$ such that $K \supset I_x$ and such that it still holds $\{\partial_j f(x) : j \notin K\} = \mathbb{R}^p$. Therefore, these p vectors are linearly independent. Because U^+ is a regular neighborhood of x we have $I_z \subset I_x$ for every $z \in U^+$; thus, $z \in \partial U^+$ implies $z^K \in \partial \mathbb{R}_+^{q-p}$. We see all conditions on K stated in the previous paragraph are satisfied.

Condition (b) gives a $W^+ \subset U^+$, $x \in W^+$ such that the situation when $z \in \partial W^+$ and $f(z) \notin \partial V^+$ simultaneously never happens. Therefore, we may just choose K as any set of $q-p$ indices such that the vectors $\{\partial_j f(x) : j \notin K\}$ are linearly independent. Such a set always exists as $\mathrm{d}f_x$ is surjective.

Condition (c) provides us with a neighborhood $W^+ \subset U^+$ of x and indexes $j_k \in I_x$ for $k = 1, \dots, d(f(x))$ such that $z^{j_k} = 0$ implies $f^{i_k}(z) = 0$ where $I_{f(z)} = \{i_1, \dots, i_{d(f(x))}\}$. Therefore, we have $\partial_j f^{i_k}(x) = 0$ for all $j \neq j_k$ but $\partial_{j_k} f^{i_k} \neq 0$ by surjectivity of $\mathrm{d}f_x$. From the same reason we see that j_k are pairwise different. As a result, the set $\{\partial_{j_k} f(x) : k = 1, \dots, d(f(x))\}$ is a set of $d(f(x))$ linearly independent vectors. Condition (c) also guarantees $\mathrm{d}f_x(C_x U^+) = C_{f(x)} V^+$ so we can choose another indices $j_k \notin I_x$ for $k = d(f(x)) + 1, \dots, p$ such that the vectors $\{\partial_{i_k} f(x) : k = d(f(x)) + 1, \dots, p\}$ are linearly independent. The union $\{\partial_{j_k} f(x) : k = 1, \dots, p\}$ of p vectors remains linearly independent since for $k > d(f(x))$ the condition $\partial_{j_k} f(x) \in$

$C_{f(x)}V^+$ implies $\partial_{j_k} f^{i_l}(x) = 0$ for all $l = 1, \ldots, d(f(x))$ whereas for $k \leq d(f(x))$ these derivatives are the only non-zero elements of $\partial_{j_k} f(x)$. We set $K = \{1, \ldots, q\} \backslash \{j_1, \ldots, j_p\}$ and check whether the requirements on K are satisfied: Firstly, the vectors $\partial_i f(x), i \notin K$ are linearly independent. Secondly, if there is a $z \in \partial U^+$ such that $z^i = 0$ for some $i \in I_z \subset I_x$ and $z^k \neq 0$ for any $k \in K$ then there must be an $l \in \{1, \ldots, p\}$ such that $i = j_l$. Because $j_l \notin I_x$ for $l > d(f(x))$ we must have $l \leq d(f(x))$; hence, $z^{j_l} = 0$ implies $f^{i_l}(z) = 0$ and $f(z) \in \partial V^+$. Therefore, both conditions of the first paragraph are satisfied and the theorem is proven. $\square$

Remark 39 (Canonical Projections). Let $\pi : \mathbb{R}^q_+ \to \mathbb{R}^p_+$ be the canonical projection defined by $\pi(z) = (z_1, \ldots, z_p)$ for $z = (z_1, \ldots, z_q) \in \mathbb{R}^q$. We see that $I_{\pi(z)} = I_z \cap \{1, \ldots, p\}$. Therefore, if $x \in \mathbb{R}^q_+$ then $\mathrm{d}\pi_x(C_x \mathbb{R}^q_+) = \pi(\{v \in \mathbb{R}^q : v^{I_x} = 0\}) = \{v \in \mathbb{R}^p : v^{I_x \cap \{1, \ldots, p\}} = 0\} = C_{\pi(x)} \mathbb{R}^p_+$ and also $\pi(\partial_i \mathbb{R}^q_+) \subset \pi(\partial_i \mathbb{R}^p_+)$ for all $i \in I_{\pi(z)}$. We see that (iii) is satisfied. Therefore, a smooth map is expressible as a projection near x if and only if it has surjective differential at x and satisfies (iii). However, (i) or (ii) do not always hold for a projection: Let $q = 3$, $p = 2$, $\pi(z_1, z_2, z_3) = (z_1, z_2)$ and $x = 0$, for example. Then $C_x \mathbb{R}^3_+ = 0$ so that (i) can not hold. Also there are points $(z_1, z_2, 0) \in \partial_i \mathbb{R}^3_+$ with $\pi(z_1, z_2, 0) = (z_1, z_2) > 0$ arbitrary close to x so that (ii) does not hold.

Theorem 22 (Local Form of an Immersion). *Let $U^+ \subset \mathbb{R}^q_+$, $V^+ \subset \mathbb{R}^p_+$ be open subspaces, $x \in U^+$, and let $f^+ : U^+ \to V^+$ be a smooth map. Let all the following conditions hold:*

(i) $\mathrm{d}f^+_x : T_x U^+ \to T_{f(x)} V^+$ is injective

(ii) The induced map $T_x U^+ / C_x U^+ \to T_{f^+(x)} V^+ / C_{f^+(x)} V^+$ is injective

(iii) There is an open neighborhood $W^+ \subset U^+$ of x and indexes $j_k \in$

$I_{f(x)}$ for every $k = 1, \dots, d(x)$ such that $f^+(\partial_{i_k} W^+) \subset \partial_{j_k} V^+$ where $I_x = \{i_1, \dots, i_{d(x)}\}$. We also require that $f^+(W^+) \subset \partial_j W^+$ for all $j \in I_{f^+(x)} \backslash \{j_1, \dots, j_{d(x)}\}$

Then there is an open neighborhood $\tilde{U}^+ \subset U^+$ of x, $\tilde{V}^+ \subset V^+$ of $f^+(x)$ such that $f^+(\tilde{U}^+) \subset \tilde{V}^+$, an open subspace $V'^+ \subset \mathbb{R}^p_+$ and a diffeomorphism $\varphi^+ : \tilde{V}^+ \to V'^+$ such that

$$\varphi^+ \circ f^+ = \iota_a\big|_{\tilde{U}^+} \tag{31}$$

where $\iota_a : \mathbb{R}^q \to \mathbb{R}^p$ is the embedding $z \mapsto (z, a)$ for some $a \in \mathbb{R}^{p-q}_+$.

Proof. We may arrange the same setting as in the proof of the previous theorem 21. If $\psi : \mathbb{R}^p \to \mathbb{R}^p$ is a map invertible near (x, a) which for $(z_1, z_2) \in \mathbb{R}^q \times \mathbb{R}^{p-q}$ near (x, a) satisfies

$$\psi(z_1, z_2) = f(z_1) + \iota_K(z_2 - a) \tag{32}$$

where $\iota_K : \mathbb{R}^{p-q} \to \mathbb{R}^p$ is the canonical inclusion to the K-coordinates, $K \subset \{1, \dots, p\}$, $|K| = p - q$, $a \in \mathbb{R}^{p-q}_+$, then $\varphi = \psi^{-1}$ clearly fulfills the Equation 31. We will show ψ satisfies conditions of the IFT 20 at (x, a) for some choice of K and a. Then we will get an open neighborhood $V'^+ \subset \mathbb{R}^p_+$ of (x, a) which is mapped diffeomorphically onto an open neighborhood $\tilde{V}^+ \subset \mathbb{R}^p_+$ of $f(x)$ by $\psi^+ = \psi\big|_{\mathbb{R}^p_+}$ and we will find $\tilde{U}^+$ just by shrinking U^+ so that $f(\tilde{U}^+) \subset \tilde{V}^+$.

For every $k = 1, \dots, d(x)$ condition (iii) implies $\partial_i f^{j_k}(x) = 0$ for all $i \neq i_k$ and also $\partial_i f^j(x) = 0$ for all i and $j \in I_{f(x)} \backslash \{j_1, \dots, j_{d(x)}\}$. Condition (ii) then gives that j_k are pairwise different and that $\partial_{i_k} f^{j_k}(x) \neq 0$ for every $k = 1, \dots, d(x)$. Indeed, we may write $T_x\mathbb{R}^q_+ / C_x\mathbb{R}^q_+ = \{v \in \mathbb{R}^q : v^i = 0 \text{ for } i \notin I_x\}$, $T_{f(x)}\mathbb{R}^p_+ / C_{f(x)}\mathbb{R}^p_+ = \{v \in \mathbb{R}^p : v^j = 0 \text{ for } j \notin I_{f(x)}\}$, and we see that $\partial_{i_k} f^{j_k}(x)$ are the only possible non-zero coordinates which can make the vectors $\partial_i f(x), i \in I_x$ projected to the latter space linearly independent.

Altogether we get that $\nabla f^{j_k}(x), k = 1, \ldots, d(x)$ are $d(x)$ linearly independent vectors. Condition (i) implies we may complete $\nabla f^{j_k}(x), k = 1, \ldots, d(x)$ with another vectors $\nabla f^{j_k}(x)$, $j_k \in \{1, \ldots, p\}$, $k = d(x)+1, \ldots, q$ so that the vectors $\nabla f^{j_k}(x), k = 1, \ldots, q$ are linearly independent. By the previous construction, however, we see that $j_k \notin I_{f(x)}$ for any $k > d(x)$ since all linearly independent vectors for $j_k \in I_{f(x)}$ are already exhausted. We may now set $K = \{1, \ldots, p\} \backslash \{j_1, \ldots, j_q\}$, $a = f(x)^K$ and check the conditions:

Firstly, $\nabla f^j(x), j \notin K$ are linearly independent vectors by construction.

Secondly, we will take a neighborhood $V' = U \times \mathbb{R}^{p-q} \subset \mathbb{R}^p$ of (x, a) so that ψ is well defined on V' and check that it restricts to a map $\psi : V'^+ \to \mathbb{R}^p_+$. Let $z_1 \geq 0$, $z_2 \geq 0$. Then $\psi^j(z_1, z_2) = f^j(z_1) \geq 0$ for $j \notin K$. In order to make this to hold also for $j \in K$ we shrink V' in the following way. We find an $\varepsilon > 0$ such that $f(x)^j > \varepsilon$ whenever $f(x)^j > 0$ for some $j \in K$. We shrink V' so that for these j it holds $f^j(z_1) > \varepsilon$, $|z_2^i - f^j(x)| < \varepsilon$ for all $(z_1, z_2) \in V'$ where i is such that $a^i = f^j(x)$. Now if $f^j(x) > 0$ then $\psi^j(z_1, z_2) = f^j(z_1) + z_2^i - f^j(x) \geq \varepsilon - \varepsilon = 0$ and if $f^j(x) = 0$ then $\psi^j(z_1, z_2) = f^j(z_1) + z_2^i \geq 0$ since both summands are non-negative. Therefore, the map really restricts correctly.

Lastly, we will check that $\psi(\partial V'^+) \subset \partial \mathbb{R}^p_+$. If $(z_1, z_2) \in V'^+$ and $z_1^i = 0$ for some i then $i = i_k \in I_x$ because of regularity of U^+ and this implies $\psi^{j_k}(z_1, z_2) = f^{j_k}(x) = 0$. By shrinking of V' we may achieve that $a^i > 0$ for some i implies $z_2^i > 0$ for all $(z_1, z_2) \in V'$. There is some $j \in K$ such that $a^i = f^j(x)$ by definition and we see that $z_2^i = 0$ implies $f^j(x) = a^i = 0$. This means that $\psi^j(z_1, z_2) = f^j(z_1) + z_2^i - f^j(x) = f^j(z_1) = 0$ where the last equality holds because $j \in K \cap I_{f(x)} = I_{f(x)} \backslash \{j_1, \ldots, j_{d(x)}\}$. Therefore, $\psi(\partial V'^+) \subset \partial \mathbb{R}^p_+$ and both assumptions of the IFT are satisfied. $\square$

Remark 40 (Linear Embedding). Let $\iota_a : \mathbb{R}^q_+ \to \mathbb{R}^p_+$ for some $a \in \mathbb{R}^{p-q}_+$ be the embedding defined by $\iota_a(z) = (z, a)$ for all $z \in \mathbb{R}^q_+$. Let $x \in \mathbb{R}^q_+$. Then $(\mathrm{d}\iota_a)_x = \iota : v \mapsto (v, 0)$ is injective and the induced map $T_x\mathbb{R}^q_+/C_x\mathbb{R}^q_+ \to T_{\iota_a(x)}\mathbb{R}^p_+/C_{\iota_a(x)}\mathbb{R}^p_+$ is injective as well since $I_x \subset I_{\iota_a(x)}$. Condition (iii) is clearly satisfied too. We may conclude that a smooth map is expressible as a linear embedding ι_a for some a near x if and only if it satisfies conditions (i) to (iii).

THEOREM 23 (ODE Theorem for the Euclidean Corner). *Let $U^+ \subset \mathbb{R}^n_+$ be an open subspace and $X^+ : U^+ \to \mathbb{R}^n$ a smooth map. For $(x_0, t_0) \in U^+ \times \mathbb{R}$ consider the following initial value problem*

$$\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = X^+(\gamma(t)) \tag{33}$$
$$\gamma(t_0) = x_0$$

A solution of the initial value problem at (x_0, t_0) is a smooth curve $\gamma : I \to U^+$ where I is a non-trivial interval containing t_0 such that the differential equation is satisfied for all $t \in I$ and the initial condition holds.

The following statements hold:

(a) Every two solutions of the initial value problem at $(x_0, t_0) \in U^+ \times \mathbb{R}$ agree on their common domain.

(b) Let $\gamma : I \to U^+$ be a solution at (x_0, t_0). If there is an $i \in I_{x_0}$ such that $X^+_i(x_0) > 0$ / $X^+_i(x_0) < 0$ then t_0 is a left / right end-point of I, respectively.

Assume that X^+ satisfies the following boundary conditions for all points $x \in \partial U^+$:

(I) If $i \in I_x$ and $X^+_i(x) = 0$ then $X^+_i(x') = 0$ for all $x' \in \partial_i U^+$.

(II) If $i, j \in I_x$ such that $X^+_i(x) \neq 0$ and $X^+_j(x) \neq 0$ then $X^+_i(x)$ and $X^+_j(x)$ have the same sign.

Then the following holds:

(c) For any $(x_0, t_0) \in U^+ \times \mathbb{R}$ there is an $\varepsilon > 0$, an interval $I = (t_0 - \varepsilon, t_0 + \varepsilon)$ / $[t_0, t_0 + \varepsilon)$ / $(t_0 - \varepsilon, t_0]$ according to whether $X_i^+(x_0) = 0$ for all $i \in I_{x_0}$ / there exists an $i \in I_{x_0}$ such that $X_i^+(x_0) > 0$ / there exists an $i \in I_{x_0}$ such that $X_i^+(x_0) < 0$, there is an open neighborhood $V^+ \subset U^+$ of x_0 and a smooth map $\psi^+ : V^+ \times I \to U^+$ such that $\psi_x^+ : I \to U^+$ is a solution of the initial value problem for X at (x, t_0) for every $x \in V^+$.

Proof. We assume there is an open subspace $U \subset \mathbb{R}^n$ and a smooth map $X : U \to \mathbb{R}^n$ such that $U^+ = U \cap \mathbb{R}^n_+$ and $X\big|_{U^+} = X^+$. Let $(x_0, t_0) \in U^+ \times \mathbb{R}$. The classical ODE theorem for $\mathbb{R}^n$ (e.g. Theorem 17.9 in [4, p. 443]) asserts: There is an $\tilde{\varepsilon} > 0$, an open neighborhood $V \subset U$ of x_0 and a smooth map $\psi : V \times (t_0 - \tilde{\varepsilon}, t_0 + \tilde{\varepsilon}) \to U$ such that $\psi_x : (t_0 - \tilde{\varepsilon}, t_0 + \tilde{\varepsilon}) \to U$, $\psi_x(t) = \psi(x, t)$ is a unique solution of the initial value problem for X at (x, t_0) for every $x \in V$. Solutions of the initial value problem for X at $(x, t) \in U \times \mathbb{R}$ are defined on open intervals and agree on their common domain.

A solution $\gamma : I \to U^+$ for X^+ can be extended to a solution $\tilde{\gamma} : \tilde{I} \to U$ for X : If I is an open interval then γ is immediately a solution for X too. Assume that $t_0 \in I$ is the left end-point of I. Then we define

$$\tilde{\gamma}(t) = \begin{cases} \psi_p(t) & t \in (t_0 - \tilde{\varepsilon}, t_0] \\ \gamma(t) & t \in I, t \geq t_0 \end{cases}$$

and see it is a smooth curve $\tilde{\gamma} : \tilde{I} \to U$ which extends γ. We may similarly extend γ beyond the right end-point of I if it exists and get a curve $\tilde{\gamma}$ defined on an open interval $\tilde{I}$ containing I. This is a solution for X extending γ

From the principle above we see immediately that uniqueness of solutions for X^+ follows immediately from uniqueness of solutions for X. Therefore, condition (a) holds.

The proof of (b) is trivial : Let $X_i(x_0) > 0$ and assume there is some $\delta > 0$ such that γ is defined for $t \in (t_0 - \delta, t_0]$ as an integral curve of X^+ in U^+. However, $\dot{\gamma}_i(t_0) > 0$ and $\gamma_i(t_0) = 0$ imply $\gamma_i(t) < 0$ for all $t < t_0$ which is a contradiction. Similarly we deduce a contradiction if $X_i(x_0) < 0$.

Let us prove (c): If $x_0 \in \mathrm{Int}(U^+)$ we can find an open neighborhood $V^+ \times (t_0 - \varepsilon, t_0 + \varepsilon) \subset \psi^{-1}(\mathrm{Int}(U^+))$ of (x_0, t_0). We may take ψ^+ to be the restriction of ψ to $V^+ \times (t_0 - \varepsilon, t_0 + \varepsilon)$ and the claim follows. If $X_i(x_0) = 0$ for some $i \in I_{x_0}$ we set $W^+ = \partial_i U^+$, identify $\{x \in \mathbb{R}^n : x^i = 0\} \simeq \mathbb{R}^{n-1}$ and define a smooth function $X^+_{\hat{i}} : W^+ \subset \mathbb{R}^{n-1}_+ \to \mathbb{R}^{n-1}$ as X^+ with the i-th component omitted. Due to condition (I) a solution of Equation 33 for $X^+_{\hat{i}}$ lifts to a unique solution for X^+ by adding 0 as the i-th component. We see that in the case when $X^+_i(x_0) = 0$ for all $i \in I_{x_0}$ we may repeat this argument until we get $x_0 \in \mathrm{Int}(W^+)$ for some W^+ and the previous case applies. Assume there is an $i \in I_{x_0}$ such that $X^+_i(x_0) \neq 0$. We may suppose $X^+_i(x_0) \neq 0$ for all $i \in I_{x_0}$; otherwise, reduce the i's such that $X^+_i(x_0) = 0$ as above. By condition (II) we have either $X^+_i(x_0) > 0$ or $X_i(x_0) < 0$ for all $i \in I_{x_0}$. We will assume $X^+_i(x_0) > 0$ and proceed similarly in the other case:

Clearly, if $(x, t_0) \in U^+ \times \mathbb{R}$ and there is an $0 < \eta \leq \tilde{\varepsilon}$ such that $\psi(x, [t_0, t_0 + \eta)) \subset U^+$ then $\psi_x : [t_0, t_0 + \eta) \to U^+$ is a solution for X^+. Therefore, negation of (c) provides us for every $n \in \mathbb{N}$ with an $x_n \in U^+$ and $t_0 \leq t_n \leq t_0 + \frac{1}{n}$ such that $x_n \mapsto x_0$ but $\psi(x_n, t_n) \notin U^+$. The idea is to use continuity of ψ and restrict to a neighborhood of x_0 such that the integral curves ψ_{x_n} can not escape U^+ in an other way than through the boundary. This will give a contradiction since X is inward pointing at the boundary. Formally, let $U' \subset U$ be an open neighborhood of x_0 such that $U' \cap \mathbb{R}^n_+ \subset U^+$ is a regular neighborhood of x_0 and $X_i(x) > 0$ for all $i \in I_{x_0}$. By continuity of ψ we get an open neighborhood $W \subset U'$ of x_0 and $0 < \delta \leq \tilde{\varepsilon}$ such that $\psi(W \times (t_0 - \delta, t_0 + \delta)) \subset U'$. We assume n is large enough so that $(x_n, t_n) \in W^+ \times [t_0, t_0 + \delta)$. We set

$$s_n = \sup \left\{ t_0 \leq t \leq t_0 + \tilde{\varepsilon} : \psi(p_n, [t_0, t)) \subset U'^+ \right\}.$$

Then clearly $t_n \geq s_n$. We know that $X_i(x_n) > 0$ implies $(\psi_{x_n})^i(t) > 0$ for $t > t_0$ close to t_0. By the choice of neighborhoods it holds $X_i(x_n) > 0$ for all $i \in I_{x_n} \subset I_{x_0}$ and it follows $s_n > t_0$. If $\psi(x_n, s_n) \in \mathrm{Int}(U^+)$ then continuity of the curve $\psi_{x_n} : (t_0 - \delta, t_0 + \delta) \to U'$ at s_n gives an $s_n < s'_n < t_0 + \delta$ such that $\psi_{x_n}([t_0, s_{n'})) \subset U'^+$. This is a contradiction with the choice of s_n. If $\psi(x_n, s_n) \notin \mathrm{Int}(U^+)$ then either $\psi(x_n, s_n) \notin U^+$ or $\psi(x_n, s_n) \in \partial U^+$. In the first case, there is an $t_0 < r_n < s_n$ such that $\psi(x_n, r_n) \in \partial U^+$ by the Intermediate Value Property and in the second case we just set $r_n = s_n$. The curve $\psi_{x_n} : [t_0, r_n] \to U'^+$ might be considered as an integral curve of X^+ at $(\psi(x_n, r_n), r_n)$. Nevertheless, $X_i^+(\psi(x_n, r_n)) > 0$ holds for all $i \in I_{\psi(x_n, r_n)}$ but r_n is not a left end-point of $[t_0, r_n]$. This is a contradiction with (b). As a result, (c) holds. $\square$

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