Proseminar on Geometry of Curves

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Abstract

Curves in the Euclidean space will be studied from the point of view of differential geometry using only elementary methods of (local) analysis and linear algebra. In particular, no topology or manifold theory is assumed!

A curve is a regular smooth map from an interval to the ambient space modulo smooth reparametrizations. Besides the obvious notions of a *tangent line at a point* (local) and the *total length* (global), it is possible to define higher-order invariants like the *curvature* and *torsion*. We will do this with the help of a *Frenet frame* — a special reference frame moving along the curve which can be constructed, for a generic curve, from higher-order derivatives via the Gramm-Schmidt orthogonalization.

The *total curvature* of a *simple closed curve* is related to its invariants like the *rotation index (turning number)* in the planar case by a theorem of Hopf and the *bridge number* in the spatial case by a theorem of Milnor.

The *Four-vertex theorem* states that the curvature of a simple closed plane curve has at least four local extrema, which translates into the statement that its *evolute* has at least four *cusps*.

We will quantify how much a simple curve has to be curved if it is closed and how much if it is knotted by proving *Fenchel's theorem* and the *Fáry-Milnor theorem* which give lower bounds on the total curvatures in the respective cases. Moreover, we will see that *convex plane curves* are precisely the closed curves with the least possible total curvature.

Using the Gauß theorem (without proof) we will derive an *isoperimetric inequality* between the length of a simple closed plane curve and the area enclosed by it.

In order to prove some of the theorems above, we will need to show that it is possible to lift a function defined on a star-shaped domain from the circle to the real line and that it is possible to write the total curvature of a curve as a limit of total angles of approximating polygons. These are technical results which are interesting per se.

We will see examples of some amazing curves of practical and historical importance. The *cycloid*, for example, was used by C. Huygens in the 17th century to improve the pendulum clock in order to enable sailors in the Atlantic ocean to measure the longitude more precisely. We will explain this and also the function of the south pointing chariot, involut gear or planimeter.

Contents

1 Curves in Euclidean space

Define the notions of a (smooth regular) *parametrized curve* $c\colon I \to \mathbb{R}^n$, a *reparametrization map* $\varphi\colon \tilde{I} \to I$ and a *reparametrization* $\tilde{c} = c \circ \varphi\colon \tilde{I} \to \mathbb{R}^n$. Explain what smoothness and regularity mean on closed intervals. Define an (unparametrized) *curve* c as an equivalence class of parametrized curves $c\colon I \to \mathbb{R}^n$ under the relation "being a reparametrization of". The parametrized curve $c\colon I \to \mathbb{R}^n$ is then called a *parametrization* of the corresponding curve c. A *point on* c is an equivalence class of pairs $c\colon I \to \mathbb{R}$, $t \in I$, where two pairs $c\colon I \to \mathbb{R}^n$, $t \in I$ and $\tilde{c}\colon \tilde{I} \to \mathbb{R}^n$, $\tilde{t} \in \tilde{I}$ are equivalent if there is a reparametrization $\varphi\colon \tilde{I} \to I$ such that $\tilde{c} = c \circ \varphi$ and $\tilde{t} = \varphi(t)$.

Discuss orientations, define an *oriented curve* c and the curve with *reversed orientation* $\bar{c}$. Show that every curve corresponds precisely to two oriented curves — one being the orientation reversal of the other. Define the *image* $\mathrm{im}(c) \subset \mathbb{R}^n$ of c and what it means for a curve to be *simple*. Define the notion of a *periodic parametrization* and call a curve that admits it *closed*. Notice that in contrast to $\mathrm{im}(c)$, the period is not a property of the curve c but rather of its parametrization $c\colon I \to \mathbb{R}^n$.

Define the *tangent line* $Tc(t)$ of a parametrized curve $c\colon I \to \mathbb{R}^n$ at $t \in I$ and prove that $Tc(t) = T\tilde{c}(\varphi^{-1}(t))$ for a reparametrization $\tilde{c} = c \circ \varphi$.

Define the *total length*

$$L(c) = \int_a^b \|\dot{c}(t)\| dt$$

of a parametrized curve $c\colon [a,b] \to \mathbb{R}^n$ and prove that $L(c) = L(\tilde{c})$. Conclude that the tangent line is associated to a point on the curve c and the total length to the curve c itself.

Define what it means that $c\colon [a,b] \to \mathbb{R}^n$ is a *parametrization by arc-length* and prove that an oriented curve c always admits such a parametrization. This parametrization is unique up to reparametrizations of the form $\varphi(t) = t + t_0$ for $t_0 \in \mathbb{R}$. Explain where the name "arc-length" comes from and argue that the parametrization by arc-length can be fixed uniquely by requiring the domain of such parametrization to be $[0, L(c)]$.

Discuss the approximation of $c\colon I \to \mathbb{R}^n$ with *approximating polygonials* P determined by their vertices $c(t_0)$, …, $c(t_k)$ for some subdivision $a = t_0 < t_1 < \cdots < t_k = b$ of $[a,b]$ and $k \in \mathbb{N}$. Mention that the supremum of the combinatorially defined total length $L(P)$ over such P exists, i.e., c is *rectifiable*, and equals $L(c)$.

Illustrate the notions above with simple examples like a line, circle, helix, spiral or a graph of a function.

If there is time left:

1. Can you draw a continuous curve that can not be parametrized smoothly? Can you draw a non-constant curve that admits a smooth parametrization but not a regular one? Can two non-equivalent parametrized closed curves have the same image? (Hint: multiple covers.) What if the curves are in addition assumed to be simple?

2. Show us interesting curves of your choice like a *tractrix*, *trefoil* or *the lemniscate*. See [8] for inspiration.

Literature: [6, Section 1], [7, Sections 1 and 4], [1, Section 2.1].

2 Frenet frame and curvatures

Given a parametrized curve $c\colon I \to \mathbb{R}^n$, define the notion of a *Frenet frame* as a positively oriented orthonormal frame $e_1, \ldots, e_n\colon I \to \mathbb{R}^n$ such that $\mathrm{Lin}(e_1, \ldots, e_i) = \mathrm{Lin}(\dot{c}, \ldots, c^{(i)})$ and $e_i \cdot c^{(i)} > 0$ for all $i = 1, \ldots, n-1$. Call c a *Frenet curve* if it admits a Frenet frame. Show that c is a Frenet curve if and only if $\dot{c}(t), \ldots, c^{(n-1)}(t)$ are linearly independent for every $t \in I$, and, in this case, $e_1(t), \ldots, e_{n-1}(t)$ are obtained from $\dot{c}(t), \ldots, c^{(n-1)}(t)$ by the Gramm-Schmidt orthogonalization.

Prove the *Frenet relations*

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \vdots \\ \vdots \\ \vdots \\ \dot{e}_{n-1} \\ \dot{e}_n \end{pmatrix} = \begin{pmatrix} 0 & \omega_1 & 0 & 0 & \cdots & 0 \\ -\omega_1 & 0 & \omega_2 & 0 & \ddots & \vdots \\ 0 & -\omega_2 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & \omega_{n-1} \\ 0 & \cdots & \cdots & 0 & -\omega_{n-1} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ \vdots \\ \vdots \\ e_{n-1} \\ e_n \end{pmatrix}$$

for some smooth functions $\omega_1, \ldots, \omega_{n-1}\colon I \to \mathbb{R}$. Define the *curvatures* $\kappa_1, \ldots, \kappa_{n-1}\colon I \to \mathbb{R}$ by

$$\kappa_i(t) = \frac{\omega_i(t)}{\|\dot{c}(t)\|}.$$

Show that $\kappa_1, \ldots, \kappa_{n-2}\colon I \to \mathbb{R}$ are strictly positive functions and explain why κ_{n-1} is called the *torsion*.

Show that $e_1, \ldots, e_n$ and $\kappa_1, \ldots, \kappa_{n-1}$ transform like $\tilde{e}_i = e_i \circ \varphi$ and $\tilde{\kappa}_i = \kappa_i \circ \varphi$ under an orientation preserving reparametrization $\varphi\colon \tilde{I} \to I$. Therefore, they are properties of the oriented curve c. Consider an (orientation preserving) Euclidean motion $F(x) = Rx + b$, $x \in \mathbb{R}^n$ for some $R \in SO(n)$, $b \in \mathbb{R}^n$, and define $\tilde{c}(t) = F(c(t))$ for all $t \in I$. Show that c and $\tilde{c}$ have the same curvatures and that the Frenet frames are related by $\tilde{e}_i = Re_i$. An oriented curve admitting a Frenet frame is called a *Frenet curve*.

If there is time left: Use a mathematical software (Mathematica, Matlab, ...) to animate the Frenet frame as it moves along a curve.

Literature: [5, Section 2A].

3 Local geometry of plane curves

Explain why every (regular) oriented curve c in the plane $\mathbb{R}^2$ is a Frenet curve. Given its parametrization $c\colon I \to \mathbb{R}^2$, call $v(t) = e_1(t)$ the *unit tangent vector*, $n(t) = e_2(t)$ the *unit normal vector* and $\kappa(t) = \kappa_1(t)$ the *(signed) curvature* at $t \in I$. Show that

$$n = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} v.$$

Explain the geometric meaning of the sign of the curvature as the direction of the rotation of the tangent line. Call the points where κ changes the sign *turning points.*

Recall the Frenet equations

$$\begin{pmatrix} \dot{v} \\ \dot{n} \end{pmatrix} = \|\dot{c}\| \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} v \\ n \end{pmatrix}.$$

Show that if $\|\dot{c}\| = 1$, then

$$v = \dot{c} \quad \text{and} \quad \ddot{c} = \kappa n.$$

Show that for any parametrization $c\colon I \to \mathbb{R}^2$, it holds

$$\kappa = \frac{\det(\dot{c}, \ddot{c})}{\|\dot{c}\|^3}.$$

Show that the curvature $\kappa\colon (a,b) \to \mathbb{R}$ is a constant function if and only if $c\colon (a,b) \to \mathbb{R}^2$ is a part of a circle with radius $\frac{1}{|\kappa|}$ if $\kappa \neq 0$ or a line if $\kappa = 0$.

Given a function $\kappa\colon (a,b) \to \mathbb{R}$, write down a parametrized curve $c\colon (a,b) \to \mathbb{R}^2$ having κ as its curvature explicitly. If κ is a linear function, the resulting curve is called the *Cornu spiral (clothoid).*

Define what it means that a parametrized curve $c_1\colon I_1 \to \mathbb{R}^2$ has a *contact of order k* for some $k \in \mathbb{N}$ with a parametrized curve $c_2\colon I_2 \to \mathbb{R}^2$ at $(t_1, t_2) \in I_1 \times I_2$. Write down the Taylor series for $c\colon I \to \mathbb{R}^2$ at t_0 and argue that the Taylor polynomial of order k has a contact of order k with c at $(0, t_0)$. The first order Taylor polynomial

$$\gamma(t) = c(t_0) + t\dot{c}(t_0)$$

is called the *osculating line.* The second-order Taylor polynomial is called the *osculating parabola.* Define the *osculating circle* of $c\colon I \to \mathbb{R}^2$ at t_0 as the circle $\zeta\colon [0, 2\pi] \to \mathbb{R}^2$ having a contact of order 2 with $c\colon I \to \mathbb{R}^2$ at $(0, t_0)$. If $\kappa(t_0) \neq 0$, then

$$\zeta(t) = c(t_0) + \frac{1}{\kappa(t_0)} n(t_0) + \frac{1}{\kappa(t_0)} \big(\sin(\kappa(t_0)t) v(t_0) - \cos(\kappa(t_0)t) n(t_0)\big).$$

The osculating line, parabola and circle, seen as curves, depend only on the curve c and the point on it and not on a specific parametrization.

Illustrate the formulas and concepts above on examples of curves that we have seen before.

If there is time left:

1. Motivate the *Cornu spiral* — linearly changing curvature makes it a good candidate (in the first order) for connecting elements between segments of constant curvature (line, circle) in civil engineering.

2. Animate the osculating circle as it moves along the curve and switches from one side to another at turning points.

Literature: [5, Sections 2A and 2B]; also [6, Section 2.2], [1, Section 2.2] and [7, Sections 5 and 6].

4 The winding and turning numbers and the total curvature of a closed plane curve

Given a smooth map $f\colon I \to \mathbb{S}^1 \subset \mathbb{R}^2$, construct a smooth function $\theta\colon I \to \mathbb{R}$ such that

$$f(t) = (\cos(\theta(t)), \sin(\theta(t)))$$

for all $t \in I$ — the *angle function* — and prove its uniqueness up to translations. Given a closed parametrized curve $c\colon [a,b] \to \mathbb{R}^2$, use θ to define the *winding number* $w_x(c)$ around $x \in \mathbb{R}^2 \backslash \mathrm{im}(c)$. Define the *turning number*, or *rotation index*, as

$$r(c) = w_0(\dot{c}).$$

Show that $w_x(c)$ and $r(c)$ are integers that change their sign under the orientation reversing reparametrization $t \mapsto -t$.

Show that if $c\colon [0,L] \to \mathbb{R}^2$ is a parametrization by arc-length and $\theta\colon [0,L] \to \mathbb{R}$ an angle function for $\dot{c}\colon [0,L] \to \mathbb{R}^2$, then it holds

$$\dot{\theta}(t) = \kappa(t)$$

for every $t \in I$, and hence

$$r(c) = \frac{1}{2\pi} \int_a^b \kappa(t) dt.$$

This integral is denoted by $\kappa(c)$ and called the *total curvature* of c. How to modify the formula above so that it holds for a general parametrization?

Formulate and prove the (smooth) *Lifting Lemma* for maps $X \to \mathbb{S}^1$, where $X \subset \mathbb{R}^n$ is a *star-shaped* open subset. The proof is analogous to the proof of the existence of θ, which is a special case of the Lifting Lemma for $X = I$.

Explain what *homotopy* and *regular homotopy* mean. Use the Lifting Lemma to argue that $w_x(c)$ does not change under a homotopy of $c\colon I \to \mathbb{R}^2 \backslash \{x\}$ via maps missing x. Apply the Lifting Lemma to θ to prove that $r(c)$ does not change under a regular homotopy.

Mention the *Whitney theorem* which states that two (regular) closed plane curves are regularly homotopic if and only if they have the same turning number. Analogously, two curves are homotopic via maps missing a point if and only if they have the same winding number around that point.

Compute $w_x(c)$, $r(c)$ and the total curvature from the definition for some examples.

If there is time left:

1. Explain the function of the *south-pointing chariot:* Let $c\colon [a,b] \to \mathbb{R}^2$ be the curve traced by the center of the chariot parametrized by arc-length. Let $c_L(t) = c(t) + \lambda n(t)$ and $c_R(t) = c(t) - \lambda n(t)$ be the paths traced by the left and right wheel for some $\lambda \in (0,\infty)$, respectively. In general, curves differing by a constant multiple of their unit normal vector are called *parallel*. Show that

$$\dot{c}_R(t) - \dot{c}_L(t) = 2c\dot{\theta}(t)v(t),$$

where $\theta\colon [a,b] \to \mathbb{R}$ is the angle function for $\dot{c}$. Therefore, a statue mounted on the output wheel of a differential between the left and right wheel with gearing ratio $\frac{1}{2c}$ will point in a constant direction.

2. Prove the Whitney theorem first for "long curves" (being straight lines outside of a compact set) and then deduce it for closed curves.

Literature: [7, Section 2.2], [3, Section 12.4 and 12.5], [1, Section 2.2]; for the notion of homotopy, see, e.g., Wikipedia.

5 Double points, double tangents and turning points

Given a closed (regular) curve c in the plane, define what a (transverse) *double point*, an *inner and outer double tangent* and a *turning point* are.

Throughout the talk, assume that c is a *generic curve.* This means that small deformations (regular homotopies) were applied to it such that: there are no self-intersections except for finitely many double points, every tangent line either intersects c in precisely one point or it is a double tangent and there are finitely many of these, there are finitely many turning points. Explain how to do it and think about possible pathological cases.

Prove the *Fabricius-Bjerre formula* relating the number of inner and outer double tangents T_- and T_+, respectively, with the number of turning points I and the number of double points D:

$$T_+ - T_- = D + \frac{1}{2}I$$

Define the notion of a *cusp* and a (regular) *curve with cusps.* If c is a curve with C cusps, then the *Theorem of Ferrand* guarantees that the Fabricius-Bjerre formula remains true after adding $\frac{1}{2}C$ to the right-hand side and replacing the (unsigned) counts D, T_+ and T_- with signed counts.

Assume that c and the plane are oriented and pick $x \in \operatorname{im}(c)$ which is not a double point. The *Whitney formula* asserts that

$$r(c) = D_x + 2w_x(c),$$

where $r(c)$ is the turning number, D_x the signed count of double points (the signs depend on the choice of x) and $w_x(c)$ the winding number around x. The winding number of $x \in \operatorname{im}(c)$ is defined as the average of winding numbers of its adjacent components.

Resolve the double points and argue that $r(c)$ equals the signed count of the simple closed curves obtained in this way.

Fix a slope $\theta \in [0, 2\pi)$. Argue that $r(c)$ equals one half of a signed count of points where the tangent line has slope θ.

If there is time left:

1. Have a look at the proof of the Whitney formula in H. Whitney, *On regular closed curves in the plane.*

2. The *Lissajous curves*, known from signal analysis, are given by $L_{p,q}(t) = (\cos(pt), \sin(qt))$ for $t \in [0, 2\pi]$ and p, $q \in \mathbb{N}$. They are closed regular curves in $\mathbb{R}^2$ with two cusps if p is even. Counting vertical lines of tangency (with signs) if $p = 2k-1$ gives

$$r(L_{p,q}) = \begin{cases} 0 & \text{if } q \text{ is even,} \\ (-1)^{k+1}\gcd(p,q) & \text{otherwise.} \end{cases}$$

   In particular, there are only three regular homotopy classes of regular Lissajous curves for p, q relatively prime.

3. One can extend the notion of the turning number to curves with cusps in the obvious way such that a cusp contributes by $\pm 2\pi$. All combinatorial formulas for $r(c)$ generalize in a straightforward way by adding the signed count of cusps C.

Literature: [3, Chapter 12]

6 Hopf's theorem on the turning number and Jordan's theorem on the winding number of a simple closed plane curve

Consider an oriented simple closed plane curve c. Use the Lifting Lemma to prove *Hopf's theorem* which states that

$$r(c) \in \{\pm 1\}.$$

Sketch a proof of *Jordan's theorem* which states that for any $x \in \mathbb{R}^2 \backslash \text{im}(c)$, it holds

$$w_x(c) \in \{\pm 1, 0\},$$

where the sign depends on the orientation of c. We call

$$\mathcal{I}(c) = \{x \in \mathbb{R}^2 \backslash \text{im}(c) | w_x(c) \neq 0\} \quad \text{and}$$
$$\mathcal{E}(c) = \{x \in \mathbb{R}^2 \backslash \text{im}(c) | w_x(c) = 0\}$$

the *interior* and *exterior* of c, respectively.

If there is time left:

1. Define the notion of a *regular piecewise smooth curve* c in $\mathbb{R}^n$ and its *corners*. Define the *(unsigned) angle* $\alpha \in [0, \pi]$ at a corner. For $n = 2$, this can be refined to the *(signed) angle* $\alpha \in [-\pi, \pi]$ depending on the orientation of the plane. Generalize the notion of the total curvature for regular piecewise smooth plane curves and formulate a generalization of Hopf's theorem for them.

2. Jordan's theorem also holds more generally, namely for continuous simple closed plane curves. These generalizations can be proven by *smoothing* and reduction to the previous cases.

Literature: [7, p. 61–71]; see also [1, p. 47–52]

7 Convex plane curves

Consider the following definitions of convexity of a simple closed plane curve c:

1. $\mathcal{I}(c)$ is convex, i.e., with every pair of its points $\mathcal{I}(c)$ contains also the segment between them.
2. For every line $l \colon \mathbb{R} \to \mathbb{R}^2$, the set $l^{-1}(\mathrm{im}(c))$ is either
   (a) empty, or
   (b) a singleton, or
   (c) a union of two different points, or
   (d) a closed interval $[a, b]$ for $a < b$.
3. $\mathrm{im}(c)$ lies entirely on one side of $Tc(t)$ for every $t \in I$.
4. It holds either $\kappa(t) \geq 0$ for all $t \in I$ or $\kappa(t) \leq 0$ for all $t \in I$. In other words, there is no turning point.

Notice that 1 and 2 make sense for continuous simple closed curves—they are topological definitions. This is in contrast to 3 and 4, which require the notions of a tangent line and curvature, which are available only for regular curves—they are geometric definitions. Prove 1 $\iff$ 2, 3 $\iff$ 4 and 2 $\iff$ 3. In particular, all definitions are equivalent.

Let c be a simple closed plane curve parametrized by arc-length. Define the *total absolute curvature* by

$$|\kappa|(c) = \int_a^b |\kappa(t)| dt.$$

How does the formula look like for an arbitrary parametrization? Use Hopf's theorem to prove that

$$|\kappa|(c) \geq 2\pi$$

with the equality if and only if c is convex. This gives another characterization of convex simple closed curves as those minimizing the total absolute curvature.

If there is time left:

1. Give an example of a closed convex curve that is not simple. Prove that a closed convex plane curve is simple if and only if $r(c) \in \{\pm 1\}$.
2. Define the notion of *lines of support with slope* θ of a closed plane curve c as a pair of lines with slope θ which touch the curve and the curve lies entirely between them (they are tangents in the smooth case). The distance between the lines $d(\theta)$ is called the *width* of c. Prove that if $d \colon [0, \pi) \to [0, \infty)$ is a constant function, then the curve is convex.

Literature: [5, p. 30–32], [7, p. 75–79], [1, p. 52–55]

8 The Four vertex theorem for a convex simple closed plane curve

A *vertex* of a parametrized (regular) curve $c\colon I \to \mathbb{R}^2$ is a point $t \in I$ such that

$$\dot{\kappa}(t) = 0.$$

Explain the geometric meaning of a vertex as a local extremum of curvature and prove that it is independent of parametrization.

The *Four vertex theorem* states that every simple closed plane curve c has at least 4 vertices. Prove it for convex simple closed plane curves.

If there is time left:

1. Compare various proofs of the Four vertex theorem and think of the role of the convexity assumption and a way to remove it.
2. Think of a way to prove that a curve with constant width has at least six vertices.
3. How to generalize the notion of a vertex to higher dimensions?

Literature: [7, p. 72–75], [1, p. 57–61]; see also [5, p. 33]

9 Evolute and Involute

Let $c\colon I \to \mathbb{R}^2$ be a (regular) parametrized curve with nowhere vanishing curvature. Define the *evolute* $E(c)$ and the *involute* $I(c)$ by

$$E(c)(t) = c(t) + \frac{1}{\kappa(t)} n(t) \quad \text{and} \quad I(c)(t) = c(t) - L(t)v(t)$$

for all $t \in I$, where $L = \int \|\dot{c}(s)\| ds$. Therefore, $E(c)$ is a smooth curve $I \to \mathbb{R}^2$ and $I(c)$ a one-parametric family of such curves. Argue that $E(c)$ and $I(c)$ transform well under orientation preserving reparametrization and hence are associated to the oriented curve c. Use a parametrization of c by arc-length to show that

$$\begin{aligned} v_I(t) &= -\varepsilon_1 n(t) & \qquad v_E(t) &= -\varepsilon_2 n(t) \\ n_I(t) &= \varepsilon_1 v(t) & \qquad n_E(t) &= \varepsilon_2 v(t), \end{aligned}$$

where $\varepsilon_1 = \operatorname{sgn}(L\kappa)$ and $\varepsilon_2 = \operatorname{sgn}(\dot{\kappa})$. Compute the curvatures

$$\kappa_I = \frac{\|\dot{c}\|}{\|\dot{I}\|}\kappa = \frac{\operatorname{sgn}(\kappa)}{|L|} \quad \text{and} \quad \kappa_E = \frac{\|\dot{c}\|}{\|\dot{E}\|}\kappa = \frac{\|\dot{c}\|}{|\dot{\kappa}|}\kappa^3.$$

We see that $I(c)$ consists of curves that are regular on the entire I, whereas $E(c)$ is a regular curve outside of vertices of c only. Vertices where $\dot{\kappa}$ changes the sign correspond to cusps of $E(c)$. Every two curves from $I(c)$ are parallel.

Recall the geometric meaning of $E(c)(t)$ as the center of the osculating circle of c at t. For every $z \in I$, define $L_z(t) = \int_z^t \|\dot{c}(s)\| ds$ and $I_z(c)(t) = c(t) - L_z(t)v(t)$.

Show that $I_z(c)$ is precisely the curve traced by the endpoint of a string that unwinds from c starting at z. Show that

$$E(I(c)) = c \quad \text{and} \quad c \in I(E(c)).$$

Therefore, E can be seen as a derivative and I as an integral on a certain class of curves.

Define the *cycloid* $\zeta(t) = (t - \sin(t), 1 - \cos(t))$ for $t \in \mathbb{R}$ and the *inverted cycloid* $\tilde{\zeta} = i \circ \zeta$, where $i(x, y) = (x, -y)$. Prove that (draw picture)

$$\begin{aligned} E(\zeta)(t) &= \zeta(t - \pi) + (\pi, -2), \\ E(\tilde{\zeta})(t) &= \tilde{\zeta}(t - \pi) + (\pi, 2), \\ I_\pi(\zeta)(t) &= \zeta(t - \pi) + (\pi, 2), \\ I_\pi(\tilde{\zeta})(t) &= \tilde{\zeta}(t - \pi) + (\pi, -2). \end{aligned}$$

Therefore, ζ and $\tilde{\zeta}$ would correspond to $e^{\pm t}$ under the analogy with derivatives and integrals.

Show that $\tilde{\zeta}$ is a *tautochrone* — if a ball is put at $z \in \operatorname{im}(\tilde{\zeta})$ and let roll freely on $\tilde{\zeta}$ in a constant downward-pointing gravity field, then the time T in which it reaches the bottom of $\tilde{\zeta}$ does not depend on z. Conversely, every tautochrone is a part of the inverted cycloid.

The tautochrone was discovered by C. Huygens in the 17. century with the goal to improve the pendulum clock so that it can be used to measure time on ships more reliably. Explain why it is important that the pendulum is constrained to a tautochrone when the ship swings on big waves. Explain how the knowledge of time is used in nautical navigation. Explain how such an improved pendulum is constructed by looking for the evolute of the tautochrone.

If there is time left:

1. Explain the function of the *involute gear:* Consider two gearwheels with centers O_1 and O_2 whose shape is specified by closed curves c_1 and c_2, respectively. Suppose that the gearwheels rotate with angular velocities $\omega_1(t)$ and $\omega_2(t)$ and let $R_1(t)$ and $R_2(t)$ denote the corresponding rotation matrices, respectively. Suppose that the curves are parametrized such that $K(t) = R_1(t)c_1(t) = R_2(t)c_2(t)$ is a first-order contact for all t. This implies that R_1c_1 and R_2c_2 have a common unit normal $n_K(t)$ at $K(t)$ and that the orthogonal projections of the speed vectors

   $$v_1(t) = \frac{d}{ds}_{s=t} R_s c_1(t) \quad \text{and} \quad v_2(t) = \frac{d}{ds}_{s=t} R_2(t) c_2(t)$$

   onto $n_K(t)$ have the same magnitude $v(t)$. For every time t imagine a pair of circles with radii $r_1(t)$ and $r_2(t)$ centered at O_1 and O_2, respectively, which are specified by the condition that their common tangent is $n_K(t)$. The well-known relation of the angular and tangent velocity then implies that $r_1(t)\omega_1(t) = v(t) = r_2(t)\omega_2(t)$. Let $P(t)$ denote the intersection point of $n_K(t)$ and the line connecting O_1 and O_2. For an optimal function of the gearing, it is required that the ratio of $\omega_1(t)$ and $\omega_2(t)$ is constant for all times. This is equivalent to the ratio of $r_1(t)$ and $r_2(t)$ being constant and, by similarity of triangles, to the point $P(t)$ being constant. This is achieved in particular when $n_K(t) = n_K$ is constant. This condition implies that the normals of $c_1(t)$ and $c_2(t)$ have to be $R_1^{-1}(t)n_K$ and $R_2^{-1}(t)n_K$ for all times, respectively. Therefore, c_1 and c_2 are involutes of the circles (O_1, r_1) and (O_2, r_2) whose common tangent is n_K.

2. Let $\mathcal{F} = \{\mathcal{F}_s | s \in I\}$ be a family of curves parametrized by an interval. An *envelope* of $\mathcal{F}$ is a (smooth) curve $e\colon I \to \mathbb{R}^2$ such that $e(s)$ lies on $\mathcal{F}_s$ and the tangent lines of e and $\mathcal{F}_s$ at this point agree. Show that if $\mathcal{F}_s = \{(x,y) \in \mathbb{R}^2 | F(x,y,s) = 0\}$ for a smooth function $F\colon \mathbb{R}^2 \times I \to \mathbb{R}$, then an envelope is a solution of the equations

   $$F(x(s), y(s), s) = 0 \quad \text{and} \quad \frac{\partial}{\partial s} F(x(s), y(s), s) = 0.$$

   This can be interpreted by saying that e traces those points where infinitesimally close curves intersect. If these curves are light rays, then such points are brighter (an intersection point seems like a source of ray for the observer) and create patterns named *caustics*.

   Show that if $\mathcal{F}_s$ is the family of normal lines of a curve c with $\kappa \neq 0$ and $\dot\kappa \neq 0$, then the envelope $e\colon I \to \mathbb{R}^2$ exists and is equal to $E(c)$. Note that one can take $F(x,y,s) = ((x,y) - c(s)) \cdot v(s)$ as the generating function for $\mathcal{F}_s$ in this case. Conclude that any curve c is equal to the envelope of the normal vectors to $I(c)$.

Literature: [7, p. 101–111]

10 Local geometry of space curves

When is an oriented space curve c a Frenet curve? Given a parametrization $c\colon I \to \mathbb{R}^3$ and the Frenet frame e_1, e_2, e_3, call $v(t) = e_1(t)$ the *unit tangent vector*, $n(t) = e_2(t)$ *the unit normal vector* and $b(t) = e_3(t)$ the *unit binormal vector* for $t \in I$. Define the (positive) *curvature* $\kappa(t) = \kappa_1(t)$ and the *torsion* $\tau(t) = \kappa_2(t)$. Show that

$$b = v \times n.$$

Recall the Frenet equations

$$\begin{pmatrix} \dot v \\ \dot n \\ \dot b \end{pmatrix} = \|\dot c\| \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} v \\ n \\ b \end{pmatrix}.$$

Show that if $\|\dot c\| = 1$, then

$$v = \dot c \quad \text{and} \quad \kappa = \|\ddot c\|.$$

Show that for any parametrization $c\colon I \to \mathbb{R}^3$, it holds

$$\kappa = \frac{\|\dot c \times \ddot c\|}{\|\dot c\|^3} \quad \text{and} \quad \tau = \frac{\det(\dot c, \ddot c, \dddot c)}{\kappa^2}.$$

Notice that using the equations above the curvature $\kappa(t)$ can be defined for any (regular) curve, i.e., not necessary a Frenet curve, and the torsion can be defined whenever $\kappa(t) \neq 0$.

Prove that a Frenet curve $c\colon I \to \mathbb{R}^3$ is confined to a plane $L \subset \mathbb{R}^3$ if and only if $\tau(t) = 0$ for all $t \in I$. In this case, $\kappa(t) = |\tilde{\kappa}(t)|$, where $\tilde{\kappa}$ is the curvature of the plane curve $\tilde{c}\colon I \to L \simeq \mathbb{R}^2$.

Show that a space curve has constant non-zero curvature and torsion if and only if it is a part of a helix.

Let $c\colon I \to \mathbb{R}^3$ be a parametrized curve and $t_0 \in I$. Define the *osculating plane* at t_0 as the plane spanned by $\mathrm{Lin}(v(t_0), n(t_0))$ and the *osculating sphere* at t_0 as the sphere with center

$$c(t_0) + \frac{1}{\kappa(t_0)} n(t_0) - \frac{\dot{\kappa}(t_0)}{\tau(t_0)\kappa^2(t_0)} b(t_0)$$

and radius $\frac{1}{\kappa(t_0)}$ provided that $\kappa(t_0) \neq 0$ and $\tau(t_0) \neq 0$.

Consider the Taylor polynomial of order 3 of $c\colon I \to \mathbb{R}^3$ at t_0 and rewrite it in terms of v, n and b. Consider the distance function and argue that the osculating plane and sphere are unique plane and sphere with contacts of orders 2 and 3 with c at t_0, respectively.

In the setting above, the plane $\mathrm{Lin}(n(t_0), b(t_0))$ is called the *normal plane* and the plane spanned by $\mathrm{Lin}(v(t_0), b(t_0))$ the *rectification plane* at t_0. Explain that the projections of the Taylor polynomial to the osculating, normal and rectification planes have types of a *parabola*, *Niels parabola* and *cubic parabola*, respectively.

If there is time left:

1. For $t_0 \in I$, consider the linear projection $\pi_{t_0}\colon \mathbb{R}^3 \to \mathrm{Lin}(v(t_0), n(t_0)) \simeq \mathbb{R}^2$ along $b(t_0)$. Then the plane curve $\tilde{c}(t) = \pi_{t_0}(c(t))$ has the same curvature at t_0 as c.

2. Let $c\colon I \to \mathbb{R}^3$ be a *spherical curve*, i.e., it holds $\|c(t)\| = 1$ for all $t \in I$. Let $J = \det(c, \dot{c}, \ddot{c})$. Then c is a Frenet curve with

$$\kappa = \sqrt{1 + J^2} \quad \text{and} \quad \tau = \frac{\dot{J}}{1 + J^2}.$$

Literature: [5, Sections 2B and 2C], [7, p. 42–48]

11 Approximating the total curvature of a space curve with the total angle of a polygon

Define the *total angle* $\kappa(P)$ of a polygon P in $\mathbb{R}^n$ (it is the total curvature of P when seen as a regular piecewise smooth curve). Let $c\colon [a, b] \to \mathbb{R}^3$ be a closed (regular smooth) parametrized space curve. Define its total curvature $\kappa(c)$ and prove that

$$\kappa(c) = \sup_P \kappa(P),$$

where the supremum is taken over approximating polygons P for c. You might also want to prove the following fact mentioned in the first talk:

$$L(c) = \sup_P L(P).$$

If there is time left:

1. Can you generalize the result to other dimensions $n \in \mathbb{N}$?

2. Can you generalize the result by allowing c to be regular piecewise smooth?

Literature: [1, p. 72–81 and 34–37]

12 Bridge number and the total curvature of a closed space curve

Let $c\colon [a,b] \to \mathbb{R}^3$ be a closed parametrized space curve. Define the *crookedness* (*bridge number* in [1]) by

$$\mu(c) = \min_{e \in \mathbb{S}^2} \mu(c,e),$$

where $\mu(c,e)$ counts the local maxima of the function $t \to c(t) \cdot e$. Prove that the definition is invariant under reparametrizations and that $\mu(c,e) = \mu(c,-e)$. Show that if c is a convex plane curve, then $\mu(c) \in \{1, \infty\}$, and if in addition κ is nowhere vanishing, then $\mu(c) = 1$. Recall the notion of the total curvature $\mu(c)$ and prove that

$$\frac{1}{A[\mathbb{S}^2]} \int_{\mathbb{S}^2} \mu(c,e) dA[e] = \frac{1}{2\pi} \kappa(c),$$

where $A[\mathbb{S}^2] = 4\pi$ is the area of $\mathbb{S}^2$ and the symbol $\int_{\mathbb{S}^2} \mu(c,e) dA[e]$ denotes the integration of the function $e \to \mu(c,e)$ over $\mathbb{S}^2$. For our purposes, we can define this integral in spherical coordinates as the Lebesgue integral

$$\int_{\mathbb{S}^2} \mu(c,e) dA[e] = \int_0^{\pi} \int_0^{2\pi} \mu(c,\theta,\phi) \sin(\theta) d\theta d\phi,$$

where $\mu(c,\theta,\phi) = \mu(c,e)$ for $e = (\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta))$ and $\theta \in [0,\pi]$, $\phi \in [0,2\pi)$ and $dA[e]$ is replaced by $\sin(\theta) d\theta d\phi$. Conclude that

$$\kappa(c) \geq 2\pi\mu(c).$$

If there is time left:

1. Show that the crookedness of a convex plane curve equals 1.

2. Have a look at W. Milnor, *On the total curvature of knots*. Crookedness is defined for closed curves in $\mathbb{R}^n$ and the relations to the total curvature $\mu(c)$ (defined using κ_1) still hold. It is also proven that

$$\inf_{c' \in [c]} \kappa(c') = 2\pi \min_{c' \in [c]} \mu(c'),$$

   where the infimums are taken over all curves c' ambiently isotopic to c. It is proven using an approximation by polygons.

3. The number $\mu(c)$ is called *Brückenzahl* in [1]. This can be motivated as follows. Given a simple closed space curve $c\colon [a,b] \to \mathbb{R}^3$, we can project it orthogonally onto a plane $L \subset \mathbb{R}^3$ and obtain a smooth curve $c_L\colon [a,b] \to L \simeq \mathbb{R}^2$. By taking a generic L, we may assume that c_L

is a *regular projection* of c, which means that the only self-intersections of c_L are double points. Consider a double point $x = c_L(t_1) = c_L(t_2)$. Depending on the order of t_1 and t_2 and the height of $c(t_1)$ and $c(t_2)$ above L, we can distinguish whether a segment of c_L going through x is an *overcrossing* or an *undercrossing*. The curve c_L together with this information is called a *knot diagram* of c. A *bridge* is an *arc*, i.e., a segment of c_L with no undercrossing, with at least one overcrossing. Let $b(c, L)$ denote the number of such bridges in a knot diagram determined by c and L. Then $b(c) = \min_L b(c, L)$, where the minimum is taken over all regular projections, is called the *bridge number* of c. It holds

$$\min_{c' \in [c]} \mu(c') = \min_{c' \in [c]} b(c'),$$

where the minima are taken over all curves c' ambiently isotopic to c. Note, however, that $b(c)$ might not be equal to $\mu(c)$ for a single curve.

Literature: [1, p. 81–86]

13 Fenchel's Theorem on the minimal curvature of a simple closed space curve

Prove *Fenchel's theorem* which states that the total curvature $\kappa(c)$ of a simple closed space curve c satisfies

$$\kappa(c) \geq 2\pi$$

with the equality if and only if c is a convex plane curve.

Fenchel's theorem holds for any closed curve c. It also holds in any dimension $n \in \mathbb{N}$ (for $n = 2$, we mean $|\mu|(c)$). Discuss the proof of these generalizations.

If there is time left:

1. Compare various proofs in the literature (using crookedness or not).
2. Does Fenchel's theorem hold also for closed regular piecewise smooth curves? You can motivate this question by considering the *teardrop* or a polygon.

Literature: [1, p. 86–87], [5, p. 33–35], [7, p. 78–81]

14 Fundamental theorem of local curve theory and the classification of curves with constant curvatures

Formulate the *Fundamental theorem of local curve theory:* for given functions $\kappa_1, \ldots, \kappa_{n-2} \colon (a, b) \to (0, \infty)$, $\kappa_{n-1} \colon (a, b) \to \mathbb{R}$, there is a curve $c \colon (a, b) \to \mathbb{R}^n$ with curvatures $\kappa_1, \ldots, \kappa_{n-1}$ and this curve is unique up to Euclidean motions. Prove the theorem by rewriting the Frenet equation as a linear differential equation for $n \times n$ functions — components of the Frenet frame $e_1, \ldots, e_n$ —

and applying the result on the existence and uniqueness of a global solution of such an equation. The curve c can then be obtained by integrating e_1.

Assuming that the curvatures $\kappa_1, \ldots, \kappa_{n-1}$ are constant, write down a solution in terms of the matrix exponential $\exp(tK)$ of the *curvature matrix*

$$K = \begin{pmatrix} 0 & \omega_1 & 0 & 0 & \ldots & 0 \\ -\omega_1 & 0 & \omega_2 & 0 & \ddots & \vdots \\ 0 & -\omega_2 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 & \omega_{n-1} \\ 0 & \ldots & \ldots & 0 & -\omega_{n-1} & 0 \end{pmatrix}.$$

Because K is antisymmetric, it can be transformed to its normal form by conjugation with a rotation matrix $R \in SO(n)$,

$$\tilde{K} = RKR^t = \begin{pmatrix} B_1 & 0 & \ldots & 0 \\ 0 & B_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & B_k \end{pmatrix},$$

where $k = \lfloor n/2 \rfloor$ and

$$B_i = \begin{cases} \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix} & \text{for } i \in \{1, \ldots, k-1\} \text{ and } i = k \text{ for } n \text{ even}, \\ \begin{pmatrix} 0 & \lambda_k & 0 \\ -\lambda_k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{for } i = k \text{ and } n \text{ odd}, \end{cases}$$

where $\lambda_i \in [0, \infty)$ are such that $\pm i\lambda_i$ are eigenvalues of K. The exponential of $\tilde{K}$ can then be computed explicitly. Carry on the computation in the cases $n = 2$ and $n = 3$ in details and arrive at the classification of curves with constant curvatures which we have seen before.

Identify all simple closed curves with constant curvature for $n = 4$ lying in the *Clifford torus* $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4$. The standard embedding $\mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^3$ then gives the *torus knots* $T_{p,q}$ in $\mathbb{R}^3$ for p, $q \in \mathbb{N}$ relatively prime. Draw $T_{2,3}$ (two times around the equator and three times around the meridian) and identify it with the trefoil knot.

If there is time left:

1. Use mathematical software to plot torus knots conveniently.
2. Can you compute the total curvature $\kappa(T_{p,q})$ and the crookedness $\mu(T_{p,q})$ and check that they satisfy $\kappa(T_{p,q}) \geq 2\pi\mu(T_{p,q})$?

Literature: [5, Section 2D]

15 Knotted curves and the Fáry-Milnor Theorem

Define the notion of an *isotopy* of $\mathbb{R}^n$ and what it means for two curves to be *ambiently isotopic*. Here we consider continuous curves and isotopies. Show that "being ambiently isotopic" is an equivalence relation on curves. Define a *knot* as an equivalence class of simple closed curves with respect to this relation.

Argue that there is only one knot in $\mathbb{R}^2$ — the *unknot* — represented by the unit circle. The unknot in $\mathbb{R}^n$ is defined as the knot which is equivalent to the unknot in one (and hence any) plane in $\mathbb{R}^n$. Argue that the unknot is the only smooth knot in $\mathbb{R}^n$ for $n \geq 4$. A simple closed curve $c\colon I \to \mathbb{R}^3$ is called *knotted* if it is not equivalent to the unknot. Otherwise, it is called *unknotted*.

Prove the *Fáry-Milnor Theorem:* A knotted (smooth regular) simple closed space curve c satisfies

$$\kappa(c) \geq 4\pi.$$

Construct curves that are not knotted and have arbitrary large total curvature. Hint: use helix. Therefore, knowing only the total curvature, we can conclude that c is unknotted provided that $\kappa(c) < 4\pi$, but we can not show that it is knotted.

If there is time left:

1. Check the Fáry-Milnor Theorem for torus knots (numerical computation for a couple of examples including the trefoil knot is enough).

2. Torus knots are a family of non-equivalent knots. Another family of non-equivalent knots which are not equivalent to torus knots is *Lissajous knots*, which project onto Lissajous curves. Famous is also the *figure-eight knot*, which every sailor has to know, and which does not belong to either of these families. *Knot theory* aims to construct invariants (integers, polynomials, or certain algebraic structures) that can distinguish knots from each other. The bridge number b, defined as the minimum of bridge numbers $b(c)$ over isotopic curves c, is one of these. It distinguishes the trefoil knot with $b = 2$ from the unknot with $b = 0$. However, it does not distinguish the trefoil knot from the figure-eight knot. Note that by picking a parametrization and computing the total curvature, we get an upper bound $\kappa(c)/2\pi$ on the bridge number of the corresponding knot.

Literature: [1, p. 87–91]

16 Isoperimetric inequality

Let $c : [0, L] \to \mathbb{R}^2$ be a simple closed plane curve with rotation number $r(c) = 1$ parametrized by arc-length, and let $u : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable map (a vector field). The following equalities are called the *Gauß divergence theorem* and *Green's rotation theorem*, respectively:

$$\int_{\mathcal{I}(c)} (\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}) dxdy = -\int_0^L u(c(t)) \cdot n(t) dt,$$

$$\int_{\mathcal{I}(c)} (\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}) dxdy = \int_0^L u(c(t)) \cdot v(t) dt.$$

Show that these theorems are equivalent (the dimension is 2) and motivate their proofs by considering the flux of u through an infinitesimal square. Note that the minus sign in front of the integral comes from the fact that the normal vector $n(t)$ points in the interior of $\mathcal{I}(c)$, which is due to the assumption on $r(c)$. Use one of these theorems to deduce the following expression for the *area* $A(c)$ *enclosed by c:*

$$A(c) := \int_{\mathcal{I}(c)} dxdy = -\int_0^L \dot{u}_1(t)u_2(t)dt = \int_0^L u_1(t)\dot{u}_2(t)dt.$$

Express $A(c)$ and $L(c)$ in terms of Fourier coefficients of c viewed as a complex valued function and prove the *Isoperimetric inequality*

$$4\pi A(c) \leq L(c)^2$$

with equality if and only if c is a circle.

If there is time left:

1. The length $L(c)$ of a simple closed plane curve c is measured by a device called an *opisometer*. It consists of a wheel of radius r which is attached to a handle and rolled along the curve. The result of the measurement is displayed on a dial which is geared to the measuring wheel, the gearing ratio being a function of r. The principle of an opisometer is obvious.

   The area $A(c)$ is measured by a slightly more complicated device called a *planimeter* (more specifically a *polar planimeter*). It consists of two solid bars of lengths l_1 and l_2 linked together by a joint which can rotate freely in the plane. One end of the linkage is fixed in the plane while the other traces once around c during the measurement. The measuring wheel of radius r is mounted on the bar whose one end traces c in the distance ε from the joint and its rotation axis is fixed parallel with the bar. The result of the measurement is displayed on a dial which is geared to the measuring wheel, the gearing ratio being a function of r, l_1, l_2 and ε.

   Apply Green's theorem to a unit vector field perpendicular to the bar at the point where it traces c and explain why $A(c)$ is proportional to the total distance d rolled by the wheel. Use the geometry of the problem to express the proportionality constant as a function of (l_1, l_2, ε).

Literature: [1, p. 61–64], [7, Section 5]

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